SOME GENERAL RESULTS ON REPRESENTATIONS

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1. INTRODUCTION

Let $P = \{P_1, P_2, P_3, \cdots\}$ be any sequence of distinct positive integers, then

$$(*) \qquad \prod_{i=1}^{\infty}\left(1 + X^{P_i}\right) = \lim_{m\to\infty}\prod_{i=1}^{m}\left(1 + X^{P_i}\right) = \sum_{n=0}^{\infty} R(n)X^n ,$$

where $R(n)$ is the number of representations of the integer n as the sum of distinct elements of P. If $P_i = 2^{i-1}$ $(i = 1, 2, \cdots)$, then $R(n) = 1$ for all $n \ge 0$. Brown [1] has shown that if $P_1 = 1$ and

$$P_{n+1} \le 1 + \sum_{i=1}^{n} P_i ,$$

then $R(n) \ge 1$ for all $n \ge 0$. Here we discuss some consequences of the condition

$$(**) \qquad P_{n+1} \ge 1 + \sum_{i=1}^{n} P_i .$$

Let $P_1 = 1$, if equality holds for each $n \ge 1$, then $P_i = 2^{i-1}$, $i \ge 1$. If for some n, the inequality holds, then $R(m) = 0$ for some $m > 0$, which we call an integer which is non-representable by P.

2. SOME GENERAL RESULTS

The condition (**) guarantees that $P_i \ne P_j$ for $i \ne j$. Further we may prove

Theorem 1. Every positive integer N which has a representation by the sum of distinct elements of P, then that representation is unique.

Proof. Clearly each P_i is its own unique representation since the sequence is strictly increasing and $P_{n+1} > P_1 + P_2 + P_3 + \cdots + P_n$. Suppose N had two different representations

$$N = \sum_{i=1}^{k} \alpha_i P_i = \sum_{i=1}^{m} \beta_i P_i ,$$

where α_i and $\beta_i = 0$ or 1 independently, with $\alpha_k = \beta_m = 1$. If $m = k$, then delete $P_m = P_k$ from each side and continue to do so step-by-step until the highest order term on the left is different from the highest order term on the right. Now assume $P_k > P_m$. This is an immediate contradiction since $P_k > P_1 + P_2 + \cdots + P_m + \cdots + P_{k-1}$, thus both representations cannot represent N. This evidently proves Theorem 1.

3. THE NON-REPRESENTABLE INTEGERS

In certain cases, the integers which cannot be represented by sequence P can be described by a suitable closed form. See [3] and [4], however, that is not the general situation.

Definition. Let M(n) be the number of positive integers less than n which cannot be represented by the sequence P.

Theorem 2. If

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i ,$$

then

$$M(P_{n+1}) = P_{n+1} - 2^n .$$

Proof. All the sums of the 2^n subsets of $\{P_1, P_2, P_3, \cdots, P_n\}$ distinct by Theorem 1. These sums are less than $P_{n+1} > P_1 + P_2 + \cdots$

$+ P_n$, thus

$$M(P_{n+1}) = (P_{n+1} - 1) - (2^n - 1) = P_{n+1} - 2^n$$

since $P_{n+1} - 1$ is the number of positive integers $< P_{n+1}$ and the empty subset yields the non-positive sum zero. In fact it is simple to prove further.

Theorem 3. $M(P_1 + P_2 + \cdots + P_n) = M(P_1) + \cdots + M(P_n)$.

Proof. $M(P_{n+1}) = P_{n+1} - 2^n$. Since $P_1 + P_2 + \cdots + P_n < P_{n+1}$, then all the integers between

$$\sum_{i=1}^{n} P_i$$

and P_{n+1} are non-representable. Thus

$$M(P_1 + P_2 + P_3 + \cdots + P_n) = (P_{n+1} - 2^n) - \left(P_{n+1} - \left(\sum_{i=1}^{n} P_i\right) - 1\right)$$

$$= P_1 + P_2 + P_3 + \cdots + P_n - (2^n - 1)$$

$$= P_1 + P_2 + P_3 + \cdots + P_n - (1 + 2^1 + 2^2 + \cdots + 2^{n-1})$$

$$= (P_1 - 2^0) + (P_2 - 2^1) + (P_3 - 2^2) + \cdots + (P_n - 2^{n-1})$$

$$= \sum_{i=1}^{n} M(P_i),$$

which concludes the proof of Theorem 3.

4. M(N) FOR REPRESENTABLE N

The main result in this section is the statement and proof of

Theorem 4. If

$$N = \sum_{i=1}^{k} \alpha_i P_i ,$$

then

$$M(N) = N - \sum_{i=1}^{k} \alpha_i 2^{i-1} ,$$

where each $\alpha_i = 1$ or 0.

Proof. Let

$$N = \sum_{i=1}^{k} \alpha_1 P_i ,$$

then $P_k \le N < P_{k+1}$. Thus

$$M(N) = (P_k - 2^{k-1}) + M(N - P_k) ,$$

by virtue

$$\prod_{i=1}^{k-1} \left(1 + X^{P_i}\right) = \sum_{n=0}^{q} R(n)X^n, \qquad q = \sum_{i=1}^{k-1} P_i .$$

In forming these polynomials, the representations using only P_1, P_2, $\cdots$, P_{k-1} are enumerated by the $R(n)$ for $n = 0$ to $n = P_1 + P_2 + \cdots + P_{k-1}$. The polynomial

$$\prod_{i=1}^{k-1} \left(1 + X^{P_i}\right),$$

which has degree $n = q$, has zeros behind this N. Thus, when the factor

$$\left(1 + X^{P_k}\right)$$

is multiplied in, the $R(n)$ between $n > P_k$ and $n = P_1 + P_2 + \cdots + P_k$ are precisely those from $n = 0$ to $n = P_1 + P_2 + \cdots + P_{k-1}$ followed by zero up to $P_k - 1$. Thus if we proceed by induction on the number of summands, we see the theorem is true for $N = P_k$. Assume for all N having a representation with precisely $k - 1$ summands is such that

$$N = \sum_{j=1}^{k-1} P_{i_j} ,$$

and

$$M(N) = \sum_{j=1}^{k-1}\left(P_{i_j} - 2^{i_j - 1}\right) = N - \sum_{j=1}^{k-1} 2^{i_j - 1} ,$$

then if

$$N = \sum_{j=1}^{k} P_{i_j}$$

then

$$\begin{aligned} M(N) &= \left(P_{i_k} - 2^{i_k - 1}\right) + M\left(N - P_{i_k}\right) \\ &= P_{i_k} - 2^{i_k - 1} + \sum_{j=1}^{k-1}\left(P_{i_j} - 2^{i_j - 1}\right) \\ &= \sum_{i=1}^{k}\left(P_{i_j} - 2^{i_j - 1}\right) = N - \sum_{i=1}^{k} 2^{i_j - 1} . \end{aligned}$$

which evidently proves the theorem by mathematical induction. This completes the proof of Theorem 4.

5. SOME GENERAL REMARKS

The foregoing theorems are applicable to a large class of sequences. The restriction

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i$$

in particular, fits $u_0 = 0$ and $u_1 = 1$, while

$$u_{n+2} = ku_{n+1} + u_n \qquad n \geq 0, \quad k \geq 2 .$$

The Pell sequence is the special case when $k = 2$.

Theorem 5. If $P_1 = 1$, $P_2 = k$, and $P_{n+2} = kP_{n+1} + P_n$ $n \geq 1$, then

$$P_{m+1} \geq 1 + \sum_{i=1}^{m} P_i .$$

It is true that, if $S_n = P_1 + P_2 + \cdots + P_n$, then

$$P_{n+2} + P_{n+1} - P_2 - P_1 + S_n = k(P_{n+1} - P_1 + S_n) + S_n .$$

From $P_{n+2} - kP_{n+1} = P_n$ and $P_2 - kP_1 = 0$, we assert

$$P_{n+1} = kS_n - P_n + P_1 = 1 + S_n + (k - 2)P_n + kS_{n-1} .$$

Since $k \geq 2$, the proof would be complete by induction provided it holds for $n = 1$, which one sees as follows:

$$P_2 = k \geq 1 + \sum_{i=1}^{1} P_1 = 2 .$$

This completes the proof of Theorem 5.

Another large family of sequences is given by $P_0 = 1$, $P_1 = 1$ and $P_{n+2} = P_{n+1} + kP_n$ for $n \geq 0$, $k \geq 2$. It is not difficult to establish

Theorem 6. If $P_1 = 1$, $P_2 = k + 1$, and, for $n \geq 0$,

$$P_{n+2} = P_{n+1} + kP_n ,$$

then

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i .$$

Proof. We proceed by induction. $P_1 = 1$ and $P_2 = k + 1$, thus $P_2 \geq 1 + 1$ for $k \geq 2$. Now assume

$$P_m \geq 1 + \sum_{i=1}^{m-1} P_i$$

for $m = 2, 3, \cdots, n$, then

$$\begin{aligned} P_{n+1} &= P_n + kP_{n-1} = P_n + P_{n-1} + (k - 1)P_{n-1} \\ &\geq P_n + P_{n-1} + \left(1 + \sum_{i=1}^{n-2} P_i\right) + (k - 2)P_{n-1} \\ &\geq 1 + \sum_{i=1}^{n} P_i + (k - 2)P_{n-1} . \end{aligned}$$

Clearly

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i$$

for $k \geq 2$, $n \geq 1$. This concludes the proof of Theorem 6.

We add a couple of more sequences to show we haven't captured them all. Let $P_n = F_{2n}$. (F_n is the n^{th} Fibonacci number.) Then, since

$$F_2 + F_4 + \cdots + F_{2n} + 1 = F_{2n+1} < F_{2n+2}$$

so that here, too,

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i .$$

So does $P_n = F_{2n-1}$, $n \geq 1$.

6. A FINAL CONJECTURE

Conjecture. Let H_1 and H_2 be distinct positive integers, sequence H, generated by $H_{n+2} = H_{n+1} + H_n$ $n \geq 1$, then condition (*) yields R(n) such that $R(H_n)$ is independent of the choice of H_1 and H_2.

REFERENCES

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