GENERALIZED ZECKENDORF THEOREM

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1. INTRODUCTION

The Zeckendorf theorem states that every positive integer can be uniquely represented as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in any given sum.

D. E. Daykin [1] proved the converse of the Zeckendorf theorem. Keller [2] generalized the Zeckendorf theorem and proved a restricted converse for monotone increasing integer sequences. Hence we generalize the Zeckendorf theorem in a different way and also get a restricted converse. This leaves two open questions as to validity of the unrestricted converse theorems.

2. THE GENERALIZED ZECKENDORF THEOREM

Theorem 1. Let $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = kU_{n+1} + U_n$ $(n \geq 0,$ $k \geq 1)$, then every positive integer N, has a unique representation in the form

$$N = \epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n,$$

where

$$\epsilon_1 = 0, 1, 2, 3, \cdots, \text{ or } k - 1$$

$$\left.\begin{array}{l} \epsilon_1 = 0, 1, 2, 3, \cdots, \text{ or } k \\ \text{If } \epsilon_i = k, \text{ then } \epsilon_{i-1} = 0 \end{array}\right\} i \geq 2 \quad .$$

First we prove two useful lemmas.

Lemma 1. (i) $U_{2n} = k\,(U_{2n-1} + \cdots + U_3 + U_1)$

(ii) $U_{2n+1} = k\,(U_{2n} + \cdots + U_2) + 1$.

Proof of the Lemma. (The proof will proceed by induction.)

$$U_1 = 1, \quad U_2 = k, \quad \text{and} \quad U_3 = k^2 + 1$$

from recurrence.

(i) $U_{2n+2} = kU_{2n+1} + U_{2n}$

$= k\{kU_{2n} + kU_{2n-2} + \cdots + kU_2 + 1\} + \{kU_{2n-1} + kU_{2n-2} + \cdots + kU_3 + kU_1\}$

$= k\{(kU_{2n} + U_{2n-1}) + (kU_{2n-2} + U_{2n-2}) + \cdots + (kU_2 + U_1) + 1\}$

$= k\{U_{2n+1} + U_{2n-1} + \cdots + U_3 + 1\}$

$= k\{U_{2n+1} + U_{2n-1} + \cdots + U_3 + U_1\}$, since $U_1 = 1$. End of proof of (i).

(ii) $U_{2n+3} = k\,U_{2n+2} + U_{2n+1}$

$= k\{kU_{2n+1} + \cdots + kU_3 + kU_1\} + k\{U_{2n} + \cdots + U_2)\} + 1$

$= k\{(kU_{2n+1} + U_{2n}) + (kU_{2n-1} + U_{2n-2}) + \cdots + (kU_3 + U_2)\} + 1 + k^2U_1$

$= k\{U_{2n+2} + U_{2n} + \cdots + U_4 + kU_1\} + 1$

$= k\{U_{2n+2} + U_{2n} + \cdots + U_4 + U_2\} + 1$, since U_1 and $U_2 = k$.

Lemma 2.

$$\begin{cases} U_{2n} - 1 = k(U_{2n-1} + \cdots + U_3) + (k - 1)U_1 \\ U_{2n+1} - 1 = k(U_{2n} + U_{2n-2} + \cdots + U_2) \end{cases} .$$

Proof of Lemma 2. Both parts follow easily from Lemma 1. We need to know the maximum admissible sum using $U_1, U_2, \cdots, U_m$, subject to the coefficient constraints of Theorem 1.

$$\begin{aligned} U_{2n} - 1 &= k(U_{2n-1} + U_{2n-3} + \cdots + U_1) - 1 \\ &= k(U_{2n-1} + U_{2n-3} + \cdots + U_3) + (k - 1)U_1 \end{aligned} .$$

Thus the maximum admissible sum using

$$U_1, \quad U_2, \quad U_3, \quad \cdots, \quad U_{2n-1}$$

is $U_{2n} - 1$. Now,

$$U_{2n+1} - 1 = k(U_{2n} + U_{2n-2} + \cdots + U_4 + U_2) .$$

Thus the maximum admissible sum using

$$U_1, \quad U_2, \quad U_3, \quad \cdots, \quad U_{2n}$$

is $U_{2n+1} - 1$, since U_2 has coefficient k, U_1 can have only coefficient zero.

Proof of the Theorem. The proof will proceed by induction. Verification for $s = 1$, $m < U_2 = k$ implies $n = n \cdot U_1$. Assume every integer $n < U_{s+1}$ has a unique admissible representation using only $U_1, U_2, U_3, \cdots U_s$. The maximum such representation has sum $U_{s+1} - 1$ by Lemma 2. Thus U_{s+1} is its own unique representation. For the representations for numbers

$$jU_{s+1} \le n' < (j + 1)U_{s+1} \qquad 1 \le j \le k - 2$$

we simply add $j\,U_{s+1}$ to the representations for $1 \le n \le U_{s+1}$ to get a unique representation. The coefficient of U_s can be k since the coefficient of $U_{s+1} < k$. In the interval

$$k\,U_{s+1} < n'' < U_{s+2} ,$$

the representations cannot contain U_s thus the greatest admissible representation uses $U_1, U_2, \cdots, U_{s-1}$ whose maximal admissible sum is $U_s - 1$. Thus we add to kU_{s+1} a unique representation for $n \le U_s - 1$. Thus we have now covered the interval $U_{s+1} < n < U_{s+2}$ and furthermore each such constructed representation is UNIQUE. The proof of the Theorem is complete by mathematical induction. END OF PROOF.

3. THE RESTRICTED CONVERSE TO THE GENERALIZED ZECKENDORF THEOREM

Definition: For fixed integer $K \ge 1$, a sequence $\{V_n\}_1^\infty$ of positive integers will be called a Zeckendorf K-basis (or briefly a K-basis) if every positive integer n has a unique representation in the form

$$n = \sum_{i=1}^{m} \epsilon_i V_i \,, \tag{1}$$

where the coefficients ϵ_i satisfy constraints

$$\begin{cases} \epsilon_1 = 0, 1, \cdots, K - 1 & \\ \epsilon_i = 0, 1, \cdots, K & \text{for} \quad i \geq 2 \\ \epsilon_{i-1} = 0 \text{ if } \epsilon_i = K & \text{for} \quad i \geq 2 \end{cases} \tag{2}$$

A representation in form (1) with coefficients satisfying (2) will be called admissible.

Lemma 3. If $\{V_n\}_1^\infty$ is a K-basis with $K \geq 2$, then $V_j \neq V_n$ for $j \neq n$, $1 \leq j$, $n < \infty$.

Proof. Obvious from uniqueness requirement. (For $K = 1$, $V_1 = V_2$, but V_1 has a zero coefficient in any admissible representation.)

Lemma 4. If $\{V_n\}_1^\infty$ is a non-decreasing K-basis, then V_n for $n \geq 2$ is characterized as the smallest positive integer not representable in admissible form using only $V_1, V_2, \cdots, V_{n-1}$.

Proof. Let N_n = smallest positive integer not capable of being represented in admissible form using only $V_1, V_2, \cdots, V_{n-1}$. If $N_n > V_n$, then V_n would have two admissible representations, thereby contradicting uniqueness. On the other hand, if $N_n < V_n$, then N_n itself would have no admissible representation (recalling $\{V_n\}$ is non-decreasing).

Theorem 2. Let $\{V_n\}_1^\infty$ be a non-decreasing K-basis with $K \geq 1$. Then defining $V_0 = 0$, we have

$$V_{n+2} = KV_{n+1} + V_n \quad \text{for} \quad n \geq 0, \; K \geq 1 \,. \tag{3}$$

Proof. Since $K = 1$ corresponds to Zeckendorf's theorem, we may confine our attention for $K \geq 2$. Then $\{V_n\}_1^\infty$ is strictly increasing by Lemma 3. Clearly $V_1 = 1$, and Lemma 4 in conjunction with the coefficient constraints (2) implies $V_2 = K$ [since $\epsilon_1 V_1$ can represent only the integers $1, 2, \cdots, K - 1$].

For fixed $K \geq 2$, let $\{U_n\}_1^\infty$ be the sequence defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = KU_{n+1} + U_n$ for $n \geq 0$. Then $V_0 = U_0$, $V_1 = U_1$, $V_2 = U_2$. Now, assume as an induction hypothesis that $V_i = U_i$ for $i = 1, 2, \cdots, n$, where $n \geq 2$. We wish to show $V_{n+1} = U_{n+1}$. Contained in the proof of the generalized Zeckendorf theorem is the fact that the smallest integer not representable by an admissible combination of $U_1, U_2, \cdots, U_n$ is U_{n+1}. Since $U_i = V_i$ for $i = 1, \cdots, n$, Lemma 2 implies $V_{n+1} = U_{n+1}$ and the theorem is established.

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REFERENCES

1. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers," J. London Math. Soc., 35 (1960), pp. 143-160.
2. Timothy J. Keller, "Generalizations of Zeckendorf's Theorem," Fibonacci Quarterly, Vol. 10 (1972), pp. 95-102.

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