GENERALIZATIONS OF ZECKENDORF'S THEOREM

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The Fibonacci numbers F_n are defined by the recurrence relation

$$F_1 = F_2 = 1 ,$$

$$F_n = F_{n-1} + F_{n-2} \qquad (n > 2) .$$

Every natural number has a representation as a sum of distinct Fibonacci numbers, but such representations are not in general unique. When constraints are added to make such representations unique, the result is Zeckendorf's theorem [1], [5]. Statements of Zeckendorf's theorem and its converse follow. (Alpha is an integer.)

Theorem. (Zeckendorf). Every natural number N has a unique representation in the form

$$N = \sum_{2}^{n} \alpha_k F_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+1} = 1$, then $\alpha_k = 0$.

Theorem. (Converse of Zeckendorf's Theorem) ([1], [3]). Let

$$\{x_n\}_1^\infty$$

be a monotone sequence of distinct natural numbers such that every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+1} = 1$, then $\alpha_k = 0$. Then

$$\{x_n\}_1^\infty = \{F_n\}_2^\infty .$$

There are generalizations of Zeckendorf's theorem for every monotone sequence $\{x_n\}_1^\infty$ of distinct natural numbers for which $x_1 = 1$. The following theorem is the first of many such generalizations.

Theorem 1. Let the numbers x_n be defined by the recurrence relation

$$x_1 = 1, \quad x_2 = a,$$
$$x_n = m_1 x_{n-1} + m_2 x_{n-2} \qquad (n > 2),$$

where $m_1 > 0$, $m_2 > 0$, and $a > 1$. Then every natural number N has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $\alpha_k \geq 0$ and if $\alpha_{k+p+1} \neq m_1$, $\alpha_{k+i} = m_1$ for $1 \leq i \leq p$.

i) and p is odd, then $\alpha_k < m_2$;
ii) p is even, and $k > 1$, then $\alpha_k \leq m_1$;
iii) p is even, and $k = 1$, then $\alpha_1 < a$.

The special case $m_1 = m_2 = 1$, $a = 2$ is Zeckendorf's theorem, and the case $m_2 = 1$, $a = m_1$ is a generalization proved by Hoggatt. (See p. 89)

Proof. We prove the existence of a representation by induction on N. For $N < x_2$, we have $N = Nx_1$. Take $N \geq x_2$ and assume representability for $1, 2, \cdots, N - 1$. Since $\{x_n\}_1^\infty$ is a monotone sequence of distinct natural numbers, any natural number lies between some pair of successive elements of $\{x_n\}_1^\infty$. More explicitly, there is a unique $n \geq 2$ such that $x_n \leq N < x_{n+1}$. First let $N < m_1 x_n$. There are unique integers m and r such that

$$N = mx_n + r ,$$

where $0 < m < m_1$ and $0 \le r < x_n$. If $r = 0$, then $N = mx_n$, whereas if $r > 0$, then the induction hypothesis shows that r is representable. Thus N is representable. Now let $N \ge m_1 x_n$. Since

$$x_{n+1} = m_1 x_n + m_2 x_{n-1}$$

for $n \ge 2$, there are unique integers m and r such that

$$N = m_1 x_n + m\, x_{n-1} + r\,,$$

where $0 \le m < m_2$ and $0 \le r < x_{n-1}$. If $r = 0$, then

$$N = m_1 x_n + m x_{n-1}\,,$$

whereas if $r > 0$, then r is representable. Thus N is representable. Now use the induction principle.

To prove the uniqueness of this representation, it is sufficient to prove that x_n is greater than the maximum admissible sum of numbers less than x_n according to constraints (i)-(iii). We prove this by induction on n. For $n = 1$, this is obviously true. Take $n > 1$ and assume that the sufficient condition is true for $1, 2, \cdots, n - 1$. From

$$\sum_{2}^{n} \{m_1 x_{2i-2} + (m_2 - 1)x_{2i-3}\} = \sum_{2}^{n} x_{2i-1} - \sum_{1}^{n-1} x_{2i-1} = x_{2n-1} - 1,$$

$$\sum_{2}^{n} \{m_1 x_{2i-1} + (m_2 - 1)x_{2i-2}\} = \sum_{2}^{n} x_{2i} - \sum_{1}^{n-1} x_{2i} = x_{2n} - a\,,$$

we obtain the identities

$$x_{2n-1} = \sum_{2}^{n} \{m_1 x_{2i-2} + (m_2 - 1)x_{2i-3}\} + 1 ,$$

(1)

$$x_{2n} = \sum_{2}^{n} \{m_1 x_{2i-1} + (m_2 - 1)x_{2i-2}\} + (a - 1)x_1 + 1 .$$

The induction hypothesis together with (1) shows that x_n is greater than the maximum admissible sum of numbers less than x_n. Now use the induction principle.

Theorem 1 can be extended to the case where the numbers x_n are defined by the recurrence relation

$$x_1 = 1, \qquad x_n = a_n (2 \le n \le q) ,$$

$$x_n = \sum_{1}^{q} m_k x_{n-k} \qquad (n > q) ,$$

where $m_1 > 0$, $m_k \ge 0$ for $1 < k < q$, $m_q > 0$, and $1 < a_n < a_{n+1}$ for $1 < n < q$. Every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $\alpha_k \ge 0$ and other constraints similar to those in Theorem 1 are added. For example, if $\alpha_{n-k+1} = m_k$ for $1 \le k < p < q$, then $\alpha_{n-p+1} \le m_p$. If $p = q$, then $\alpha_{n-q+1} < m_q$. These constraints must be modified to fit the initial conditions a_n. The proof of this extension follows that of Theorem 1 and uses the identity

$$x_{qn-r} = \sum_1^{q-1} m_k \sum_2^n x_{qi-r-k} + (m_q - 1)\sum_1^{n-1} x_{qi-r} + \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] x_{q-r-1}$$

$$+ \left[\frac{a_{q-r} - \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] a_{q-r-1} - 1}{a_{q-r-2}}\right] x_{q-r-2} + \cdots + \left(a_{q-r} - \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] a_{q-r-1} - \cdots - 1\right)$$

$$\cdot\, x_1 + 1 \qquad (0 \le r < q) .$$

Statements of two special cases and the proof of the second one follow.

Theorem. (Daykin [3]). Let the numbers x_n be defined by the recurrence relation

$$x_n = n (1 \le n \le q) ,$$

$$x_n = x_{n-1} + x_{n-q} \qquad (n > q) .$$

Then every natural number N has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+q-1} = 1$, then $\alpha_{k+i} = 0$ for $0 \le i < q - 1$.

Theorem 2. Let the numbers x_n be defined by the recurrence relation

$$x_n = (m + 1)^{n-1} (1 \le n \le q),$$

$$x_n = m \sum_1^q x_{n-k} \qquad (n > q) .$$

Then every natural number N has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k \le m$ and if $\alpha_{k+i} = m$ for $1 \le i < q$, then $\alpha_k < m$.

Proof. Following the proof of Theorem 1, we prove the existence of a representation by induction on N. For $N < x_q$, we have

$$N = \sum_{1}^{q-1} \alpha_k x_k ,$$

where $0 \le \alpha_k \le m$. Take $N \ge x_q$ and assume representability for 1, 2, $\cdots$, N - 1. There is a unique $n \ge q$ such that $x_n \le N < x_{n+1}$. Since

$$x_{n+1} = m \sum_{0}^{q-1} x_{n-k}$$

for $n \ge q$, there are unique integers p, m', and r such that

$$N = m \sum_{0}^{p-1} x_{n-k} + m'x_{n-p} + r ,$$

where $0 \le p < q$, $0 \le m' < m$, and $0 \le r < x_{n-p}$. If $r = 0$, then

$$N = m \sum_{0}^{p-1} x_{n-k} + m'x_{n-p} ,$$

whereas if $r > 0$, then r is representable. Thus N is representable. Now use the induction principle.

To prove the uniqueness of this representation, we prove that x_n is greater than the maximum admissible sum of numbers less than x_n according to the constraints by induction on n. For $1 \le n \le q$, we have

$$m \sum_{1}^{n-1} x_k = m \sum_{1}^{n-1} (m+1)^{k-1} = (m+1)^{n-1} - 1 < (m+1)^{n-1} = x_n .$$

Take $n > q$ and assume that the sufficient condition is true for $n - q$. Then

$$x_n = m \sum_{1}^{q-1} x_{n-k} + (m - 1)x_{n-q} + x_{n-q} .$$

The induction hypothesis shows that x_n is greater than the maximum admissible sum of numbers less than x_n. Now use the induction principle.

Zeckendorf's theorem can be further generalized to cases where the numbers x_n are defined by recurrence relations with negative coefficients.

Theorem 3. Let the numbers x_n be defined by the recurrence relation

$$x_1 = 1, \qquad x_2 = a,$$

$$x_n = m_1 x_{n-1} - m_2 x_{n-2} \qquad (n > 2) ,$$

where $0 < m_2 < m_1$ and $a > m_2$. Then every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $0 \le \alpha_k < m_1$ for $k > 1$, $0 \le \alpha_1 < a$, and if $\alpha_{k+p+1} = m_1 - 1$,

$$\alpha_{k+i} = m_1 - m_2 - 1$$

for $1 \le i \le p$, and

(i) $k > 1$, then $\alpha_k < m_1 - m_2$;
(ii) $k = 1$, then $\alpha_1 < a - m_2$.

The proof, which will not be given, follows that of Theorem 1 and uses the identity

$$x_n = (m_1 - 1)x_{n-1} + (m_1 - m_2 - 1)\sum_{2}^{n-2} x_i + (a - m_2 - 1)x_1 + 1.$$

The converse of Zeckendorf's theorem can be generalized to include as special cases the converses of the generalizations of Zeckendorf's theorem given so far.

Theorem 4. Let $\{x_n\}_1^\infty$ be a monotone sequence of distinct natural numbers such that every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $\alpha_k \geq 0$ and other constraints on $\{\alpha_k\}_1^n$ are added such that the representation of x_n is itself. Then $\{x_n\}_1^\infty$ is the only such sequence.

Proof. Assume the sequences $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ both satisfy the hypotheses, where

$$\{x_n\}_1^N = \{y_n\}_1^N$$

and $y_{n+1} \leq x_{N+1}$. Then y_{N+1} has a unique representation as a sum of numbers x_n, each of which in turn has a unique representation as a sum of numbers y_n, where $n \leq N$. On the other hand, y_{N+1} obviously represents itself and, thus, y_{N+1} has two representations in terms of numbers y_n. This contradicts the uniqueness of representation, and we conclude that

$$\{x_n\}_1^\infty = \{y_n\}_1^\infty \quad .$$

Theorem 4 does not include the converse of the following generalization of Zeckendorf's theorem.

Theorem (Brown [2]). Every natural number N has a unique representation in the form of

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