REPRESENTATIONS OF INTEGERS AS SUMS OF FIBONACCI SQUARES

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1. COMPLETENESS

If elements of a sequence can be selected, with each element being selected at most once, such that their sum is a given integer, then this integer is said to have a representation with respect to the sequence. A sequence of positive integers is complete if and only if every positive integer has at least one representation with respect to the sequence.

Consider the sequence of Fibonacci squares:

$$1,\ 1,\ 4,\ 9,\ 25,\ 64,\ \cdots\ .$$

Clearly this sequence is not complete as there are no representations for 3, 7, 8, 12, and infinitely many other integers. Let us now consider using two copies of the sequence of Fibonacci squares. Consider the sequence

$$1,\ 1,\ 1,\ 1,\ 4,\ 4,\ 9,\ 9,\ 25,\ 25,\ 64,\ 64,\ \cdots\ .$$

A few simple calculations will lead one to suspect that we now have a complete sequence. This can be proved using the following theorem given by Brown [1].

Theorem 1. Let $\{a_k\}$ be a non-decreasing sequence of positive integers with $a_1 = 1$. If

$$a_{n+1} \leq 1 + \sum_{k=1}^{n} a_k \quad ,$$

then the sequence $\{a_k\}$ is complete.

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Let us now define our sequence so that the notation will be similar to that used in Theorem 1. Let

$$a_{2k-1} = F_k^2, \qquad a_{2k} = F_k^2 .$$

Then we have

$$\sum_{k=1}^{2m} a_k = 2\sum_{k=1}^{m} F_k^2 = 2F_m F_{m+1}$$

$$\sum_{k=1}^{2m-1} a_k = \sum_{k=1}^{2m-2} a_k + F_m^2 = 2F_{m-1}F_m + F_m^2 = F_{2m} .$$

Theorem 2. The sequence of two of each of the Fibonacci squares is complete.

Proof. Let n be even with n = 2m. Then we must show that

$$a_{2m+1} \le 1 + \sum_{k=1}^{2m} a_k$$

or that

$$F_{m+1}^2 \le 1 + 2F_m F_{m+1} .$$

For $m \ge 1$,

$$F_{m-1} \le F_m$$

$$F_{m-1} + F_m \le 2F_m$$

$$F_{m+1} \le 2F_m$$

$$F_{m+1}^2 \le 2F_m F_{m+1}$$
$$F_{m+1}^2 \le 1 + 2F_m F_{m+1} .$$

The case for n odd is handled in a similar manner to complete the proof.

Theorem 3. Exactly one of the first six elements of the sequence $\{a_k\}$ can be deleted without loss of completeness.

This theorem is proved by showing that if one F_n^2 is deleted, with $n \ge 4$, then there is no representation for the integer $F_{n+1}^2 - 1$. Further, one shows that if any two of the first six elements are deleted, then completeness is again lost. The proof is completed by showing that if any one of the first six elements is deleted, it is still possible to find enough representations to establish a foundation for mathematical induction.

2. BASIC LEMMAS

Let

$$P(x) = \prod_{j=1}^{\infty} (1 + x^{a_j}) = \sum_{k=0}^{\infty} R(k)\, x^k ,$$

$$P_{2n}(x) = \prod_{j=1}^{2n} (1 + x^{a_j}) = \sum_{k=0}^{2F_nF_{n+1}} R_{2n}(k)\, x^k ,$$

$$P_{2n-1}(x) = \prod_{j=1}^{2n-1} (1 + x^{a_j}) = \sum_{k=0}^{F_{2n}} R_{2n-1}(k)\, x^k ,$$

where a_j is an element from our sequence. Then $R(k)$ is the number of representations of k as a sum of Fibonacci squares. Paralleling the method used by Klarner [2] we can prove the following lemmas.

Lemma 1.

(a) $$R_{2n}(k) = R_{2n}(2F_nF_{n+1} - k), \qquad 0 \le k \le 2F_nF_{n+1}$$

(b) $$R_{2n-1}(k) = R_{2n-1}(F_{2n} - k), \qquad 0 \le k \le F_{2n} .$$

Lemma 2.

(a) $$R_{2n+1}(k) = R_{2n}(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n+1}(k) = R_{2n}(k) + R_{2n}(k - F_{n+1}^2), \qquad F_{n+1}^2 \le k \le 2F_nF_{n+1}$$

(c) $$R_{2n+1}(k) = R_{2n}(k - F_{n+1}^2), \qquad 2F_nF_{n+1} + 1 \le k \le F_{2n+2} .$$

Lemma 3.

(a) $$R_{2n}(k) = R_{2n-1}(k), \qquad 0 \le k \le F_n^2 - 1$$

(b) $$R_{2n}(k) = R_{2n-1}(k) + R_{2n-1}(k - F_n^2), \qquad F_n^2 \le k \le F_{2n}$$

(c) $$R_{2n}(k) = R_{2n-1}(k - F_n^2), \qquad F_{2n} + 1 \le k \le 2F_nF_{n+1} .$$

Lemma 4.

(a) $$R_{2n}(k) = R(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n}(k) = R(2F_nF_{n+1} - k), \qquad F_{2n} + 1 \le k \le 2F_nF_{n+1} .$$

Lemma 5.

(a) $$R_{2n+1}(k) = R(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n+1}(k) = R(2F_nF_{n+1} - k) + R(k - F_{n+1}^2), \qquad n \ge 2 ,$$
$$F_{n+1}^2 \le k \le 2F_nF_{n+1}$$

(c) $$R_{2n+1}(k) = R(F_{2n+2} - k), \qquad 2F_nF_{n+1} + 1 \le k \le F_{2n+2} .$$

Lemma 6.

$$R(F_nF_{n+1} - k) = R(F_nF_{n+1} + k), \quad n \geq 2, \quad 0 \leq k \leq F_{n-1}F_{n+1} - 1 .$$

Lemma 7. For $n \geq 3$,

(a) $R(k) = 2R(k - F_n^2) + R(2F_nF_{n-1} - k), \quad F_n^2 \leq k \leq 2F_nF_{n-1}$

(b) $R(k) = 2R(k - F_n^2), \quad 2F_nF_{n-1} + 1 \leq k \leq 2F_n^2 - 1$

(c) $R(k) = R(2F_nF_{n+1} - k), \quad 2F_n^2 \leq k \leq F_{n+1}^2 - 1 .$

Lemma 7 can now be used to prove many representation theorems suggested by a table of values for $R(k)$, with $0 \leq k \leq 25{,}000$.

3. REPRESENTATION THEOREMS

Theorem 4.

$$R(F_nF_{n+1}) = 2R(F_{n-1}F_n), \quad n \geq 3 .$$

Proof. For $n \geq 3$,

$$2F_nF_{n-1} + 1 \leq F_nF_{n+1} \leq 2F_n^2 - 1 .$$

Using Lemma 7(b), we have

$$\begin{aligned} R(F_nF_{n+1}) &= 2R(F_nF_{n+1} - F_n^2) \\ &= 2R(F_n[F_{n+1} - F_n]) \\ &= 2R(F_nF_{n-1}) . \end{aligned}$$

Theorem 5.

$$R(F_nF_{n+1}) = 3 \cdot 2^{n-1} .$$

Proof. From the table of values we have

$$n = 1: \quad R(F_1 F_2) = 3 = 3 \cdot 2^0$$

$$n = 2: \quad R(F_2 F_3) = 6 = 3 \cdot 2^1$$

$$n = 3: \quad R(F_3 F_4) = 12 = 3 \cdot 2^2$$

which gives us a basis for induction. Now assume the statement holds for $n = k$. Then

$$R(F_k F_{k+1}) = 3 \cdot 2^{k-1} .$$

By Theorem 4,

$$R(F_{k+1} F_{k+2}) = 2R(F_k F_{k+1}) = 2 \cdot 3 \cdot 2^{k-1} = 3 \cdot 2^k .$$

We have shown that if the statement is true for $n = k$, then it is also true for $n = k + 1$. Therefore, by induction, the proof is complete.

In a thesis on this subject forty-four theorems such as theorems four and give were proved and another nine were suggested.

4. MAXIMUM AND MINIMUM VALUES OF R(k)

Since by Theorem 5,

$$R(F_n F_{n+1}) = 3 \cdot 2^{n-1} ,$$

we see that $R(k)$ increases without bound. However, maximum and minimum values of $R(k)$ can be found in each interval

$$F_n^2 \le k \le F_{n+1}^2 - 1 .$$

Theorem 6. For

$$F_n^2 \le k \le F_{n+1}^2 - 1 ,$$

the maximum value of $R(k)$ is $R(F_n F_{n+1})$.

This theorem is proved by induction.

Theorem 7. For

$$F_n^2 \le k \le F_{n+1}^2 - 1 ,$$

the minimum value of $R(k)$ is $R(k) = 3$, where

$$k = 2\left[1 + \sum_{i=1}^{n} F_{2i}^2\right], \qquad F_{2n}^2 \le k \le F_{2n+1}^2 - 1 ,$$

$$k = 2\sum_{i=1}^{n} F_{2i+1}^2 , \qquad F_{2n+1}^2 \le k \le F_{2n+2}^2 - 1 \quad .$$

By inspecting the table we see that three is the minimum value of $R(k)$ for all k included in the table. Lemma 7 assures us that no later values of $R(k)$ will be less than three. Induction is used to show that $R(k) = 3$ as specified above.

5. SIMPLE REPRESENTATIONS

A simple representation is a representation in which each Fibonacci square is used at most once. Since $F_1^2 = F_2^2 = 1$ we will allow two ones to be included in a simple representation. By distinct simple representations we mean representations whose elements are not identical.

$$R_1 = F_1^2 + \sum_{i=1}^{n} F_{k_i}^2$$

and

$$R_2 = F_2^2 + \sum_{i=1}^{n} F_{k_i}^2 \qquad (k_i \ge 3)$$

are taken to be the same simple representation since when the representations are actually written out we cannot distinguish between F_1^2 and F_2^2.

Theorem 8. An integer has at most one simple representation.

Proof. Let

$$I = F_{j_1}^2 + F_{j_2}^2 + \cdots + F_{j_n}^2$$

be a simple representation for I.

$$\sum_{i=1}^{j_n - 1} F_i^2 = F_{j_n - 1} F_{j_n} < F_{j_n}^2 < 1 .$$

Hence $F_{j_n}^2$ must be used in a simple representation for I. Similar arguments show that each $F_{j_i}^2$ must also be used.

Theorem 9. Every representation of $F_n F_{n+1}$ is simple.

Proof. Recall that

$$1 + 1 + 4 + 9 + \cdots + F_n^2 = F_n F_{n+1} .$$

Using our sequence there are $\binom{4}{2}$ ways to select the two ones and two ways to select each succeeding summand. Therefore, the number of simple representations is

$$\binom{4}{2} \cdot 2^{n-2} = 6 \cdot 2^{n-1} = 3 \cdot 2^{n-1} .$$

From Theorem 5, we have

$$R(F_n F_{n+1}) = 3 \cdot 2^{n-1} .$$

Note that we have $3 \cdot 2^{n-1}$ simple representations and a total of $3 \cdot 2^{n-1}$ representations for $F_n F_{n+1}$. Hence, every representation for $F_n F_{n+1}$ is simple.

As a result of Theorem 9 we have the following theorem, which may be called a Non-Four-Square Theorem.

Theorem 10. There does not exist a finite number n such that every positive integer can be represented as a sum of at most n Fibonacci squares.

6. VALUES OF m SUCH THAT $R(k) \neq m$

Using Lemma 7 and mathematical induction, it is possible to prove

$$R(k) \neq 5, \qquad R(k) \neq 7, \qquad R(k) \neq 13$$

for any positive integer k. It is suggested that there are an infinite number of integers m such that $R(k) \neq m$ for any positive integer k.

Further expansion of these ideas is contained in [3].

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 68 (1961), pp. 557-560.
2. David A. Klarner, "Representations of N as a Sum of Distinct Elements from Special Sequences," Fibonacci Quarterly, Vol. 4 (1966), pp. 289-306.
3. Roger O'Connell, "Representations of Integers as Sums of Fibonacci Squares," unpublished Master's Thesis, January 1970, San Jose State College, San Jose, California.

[Continued from page 102.]

$$N = \sum_{2}^{n} \alpha_k F_k ,$$

where $0 \leq \alpha_k \leq 1$ and if $\alpha_{k+1} = 0$, then $\alpha_k = 1$.

Zeckendorf's theorem provides the representation of N in terms of the minimum number of distinct Fibonacci numbers, and Brown's theorem provides the representation of N in terms of the maximum number of distinct Fibonacci numbers.