TABLE OF INDICES WITH A FIBONACCI RELATION

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In preparing tables of residues for indices of primitive roots the following situation was noted for the modulus 109. The primitive root, 11, has residues as shown corresponding to indices as given on the borders of the table. Thus the residue of 11 to the index 82 is 36.

RESIDUES OF POWERS OF 11 MODULO 109

	0	1	2	3	4	5	6	7	8	9
0		11	12	23	35	58	93	42	26	68
1	94	53	38	91	20	2	22	24	46	70
2	7	77	84	52	27	79	106	76	73	40
3	4	44	48	92	31	14	45	59	104	54
4	49	103	43	37	80	8	88	96	75	62
5	28	90	9	99	108	98	97	86	74	51
6	16	67	83	41	15	56	71	18	89	107
7	87	85	63	39	102	32	25	57	82	30
8	3	33	36	69	105	65	61	17	78	95
9	64	50	5	55	60	6	66	72	29	101
10	21	13	34	47	81	19	100	10	1	

It is noteworthy from the early entries of the table that each succeeding entry is the sum of the two that precede it. This relation can be verified for the entire table if the sums are taken modulo 109. Clearly this is an unusual situation for a table of this kind. The questions that come to mind are: Is this something very extraordinary? Under what conditions does a table of this type have this Fibonacci property?

Since the entries in the table are residues of successive powers of some quantity x, the conditions that must be fulfilled are two: (1) x must satisfy the relation

$$x^{n+1} \equiv x^n + x^{n-1} \pmod{p}$$

or what is equivalent presuming that $(x,p) = 1$ as must be the case for a primitive root,

$$x^2 \equiv x + 1 \pmod{p}$$

(2) x must be a primitive root modulo p.

The first condition leads to the congruence

$$(2x - 1)^2 \equiv 5 \pmod{p}$$

so that a necessary condition is that 5 be a quadratic residue of p. This means that p is a prime of the form $10n \pm 1$. The solutions of this quadratic congruence for primes of this type fulfill the first requirement. It is necessary, however, to determine whether they are primitive roots.

The results of this investigation for primes of the required form up to 300 are shown in the table below.

PRIME	SOLUTIONS	PRIMITIVE ROOTS
11	4, 8	8
19	5, 15	15
29	6, 24	
31	19, 13	13
41	7, 35	7, 35
59	34, 26	34
61	44, 18	44, 18
71	9, 63	63
79	50, 25	
89	10, 80	
101	23, 79	
109	11, 99	11, 99
131	12, 120	120
139	76, 64	

149	104, 41	41
151	28, 124	
179	105, 75	105
181	13, 169	
191	103, 79	
199	138, 62	
211	33, 179	
229	148, 82	
239	16, 224	224
241	52, 190	52, 190
251	134, 118	134
269	198, 72	198, 72
271	17, 225	255
281	38, 244	

The conclusion would seem to be that this phenomenon is not particularly uncommon and that there is a straightforward method of determining additional instances of this type.

[Continued from page 156.]

2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 47-52.
3. J. E. Walton and A. F. Horadam, "Some Properties of Certain Generalized Fibonacci Matrices," Fibonacci Quarterly, Vol. 9, No. 3, May, 1971, pp. 264-276.
4. Brother Alfred Brousseau, Problem H-8. Solution by John Allen Fuchs and Joseph Erbacher. Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 51-52.
5. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Vol. 1, No. 4, December, 1963, pp. 43-46.
6. Brother Alfred Brousseau, Problem H-52, Solution by V. E. Hoggatt, Jr. Fibonacci Quarterly, Vol. 4, No. 3, October, 1966, p. 254.
7. New book of number theory tables, to be published by the Fibonacci Association.