A CHARACTERIZATON OF THE FIBONACCI NUMBERS SUGGESTED BY A PROBLEM ARISING IN CANCER RESEARCH

LESLIE E. BLUMENSON*
Roswell Park Memorial Institute, Buffalo, New York 14203

1. INTRODUCTION

Cancerous growths consist of multiplying cells which invade the surrounding normal tissue. The mechanisms whereby the cancer cells penetrate among the normal cells are little understood and we have been investigating several mathematical models with a view to analyzing the movements of cells. In one such model it was necessary to enumerate the number of distinct ways a system of n cells could transform itself if adjacent cells are permitted to exchange position at most one time. It is shown below that this is simply the Fibonacci number F_{n+1}. From this characterization a very simple argument leads to a general identity for the F_n. No special knowledge of biology is required to follow the proofs and the words "person," "jumping bean," etc., could be substituted for "cell."

2. CHARACTERIZATION OF THE FIBONACCI NUMBERS

Consider a line of n cells

$$A_1A_2A_3 \cdots A_{n-1}A_n \tag{1}$$

and suppose during a unit of time a cell either exchanges position with one of its adjacent neighbors or remains fixed. It is assumed that each cell performs at most one exchange during this time. Let G_n be the number of possible distinct arrangements of the cells after the unit of time. Then $G_n = F_{n+1}$.

For $n > 2$ the number of distinct arrangements of (1) after unit time (G_n) is equal to the number in which A_n did not exchange (G_{n-1}) plus the number in which A_n exchanged with A_{n-1} (G_{n-2}), i.e.,

*Supported by Public Health Service Research Career Development Grant No. 5-K3-CA 34, 932-03 from the National Cancer Institute.

(2) $$G_n = G_{n-1} + G_{n-2} .$$

This is the well known recurrence relation for the Fibonacci numbers and since $G_1 = 1 = F_2$, $G_2 = 2 = F_3$ it follows from (2) by induction on n that $G_n = F_{n+1}$. We define for later use: $G_0 = F_1 = 1$, $G_{-1} = F_0 = 0$.

3. A GENERAL IDENTITY

In order to avoid as much as possible the typographical difficulties of subscripted subscripts, the notation for the Fibonacci numbers and the G_n will be modified as follows in this section:

$$G(n) = G_n, \qquad F(n) = F_n .$$

Let $N^2 = 2$ and $M_1, M_2, \cdots, M_N$ all positive integers. Let S_{N-1} be the set of 2^{N-1} (N-1)-tuples $(k_1, k_2, \cdots, k_{N-1})$ where $k_j = 0$ or 1, $j = 1, 2, \cdots, N - 1$. Then the identity is

(3) $$F\left(\sum_{j=1}^{N} M_j + 1\right) = \sum_{S_{N-1}} F(M_1 - k_1 + 1) \cdot F(M_N - k_{N-1} + 1) \prod_{j=2}^{N-1} F(M_j - k_{j-1} - k_j + 1),$$

where the sum is taken over all the (N-1)-tuples in S_{N-1}. (For N = 2 the product is defined to be 1.)

For N = 2, Eq. (3) reduces to the well known identity

(4) $$F(M_1 + M_2 + 1) = F(M_1 + 1)F(M_2 + 1) + F(M_1)F(M_2) .$$

It is, of course, possible to prove (3) directly from (4) by induction on N. However, the proof based on the characterization of Section 2 is extremely simple, and at the same time may suggest new geometric approaches for the analysis of multi-dimensional generalizations of the Fibonacci numbers.

Consider a line of

$$n = \sum_{j=1}^{N} M_j$$

cells which, as in (1) are performing the type of exchange described in Section 2. The number of distinct arrangements of this line after unit time is

$$G\left(\sum_{j=1}^{N} M_j\right) .$$

Now partition this group of cells before the exchanges occur in groups of M_1, $M_2, \cdots, M_N$ cells.

(5) $$A_1^1 A_2^1 \cdots A_{M_1}^1 A_1^2 A_2^2 \cdots A_{M_2}^2 A_1^3 A_2^3 \cdots A_{M_3}^3 \cdots A_1^N A_2^N \cdots A_{M_N}^N .$$

If during the unit of time $A_{M_1}^1$ and A_1^2 exchange with each other then there are only $G(M_1 - 1)$ possible distinct arrangements for the first group of cells. Set $k_1 = 1$ if these two cells do exchange and $k_1 = 0$ if they do not exchange. Then there are $G(M_1 - k_1)$ possible distinct arrangements for the first group of cells. Similarly, define $k_2 = 1$ if $A_{M_2}^2$ and A_1^3 exchange, $k_2 = 0$ otherwise. Then there are $G(M_2 - k_1 - k_2)$ possible distinct arrangements for the second group of cells. Thus for each of the four possible values of the pair (k_1, k_2) the number of distinct rearrangements of the first two groups of cells considered as a whole is

$$G(M_1 - k_1) + G(M_2 - k_1 - k_2) ,$$

and the total number of distinct rearrangements for the two groups combined is

(6) $$\sum_{k_1=0}^{1} \sum_{k_2=0}^{1} G(M_1 - k_1) + G(M_2 - k_1 - k_2) .$$

[Continued on page 292.]