A PROOF OF GOULD'S PASCAL HEXAGON CONJECTURE

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The binomial coefficients

$$B_1 = \binom{n-1}{k-1}, \quad B_2 = \binom{n-1}{k}, \quad B_3 = \binom{n}{k+1}, \quad B_4 = \binom{n+1}{k+1}, \tag{1}$$
$$B_5 = \binom{n+1}{k}, \quad B_6 = \binom{n}{k-1}$$

form a regular hexagon in the Pascal triangle. The identity

$$B_1B_3B_5 = B_2B_4B_6 \tag{2}$$

of Verner E. Hoggatt, Jr., and Walter Hansell [1] has inspired a number of results including Henry W. Gould's remarkable conjecture that

$$\gcd(B_1, B_3, B_5) = \gcd(B_2, B_4, B_6) \tag{3}$$

for all integers k and n with $0 < k < n$. Gould also had evidence of analogous results including the similar formula for the Fibonomial coefficients

$$\left\{ {m \atop r} \right\} = F_m F_{m-1} \cdots F_{m-r+1} / F_1 F_2 \cdots F_r , \tag{4}$$

in which F_n is the n^{th} Fibonacci number. (See [2] and [3].)

In this paper, we prove a generalized Gould hexagon theorem that includes (3) and the analogous property for the Fibonomial coefficients $\left\{ {m \atop r} \right\}$.

Let $a_1, a_2, a_3, \cdots$ be a sequence of nonzero integers such that both

$$\gcd(a_m, a_n) \mid a_{m+n} \tag{5}$$

and

$$\gcd(a_m, a_{m+n}) \mid a_n \tag{6}$$

for all m and n in $Z^+ = \{1, 2, 3, \cdots\}$. Let

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} m \\ r \end{bmatrix} = a_m a_{m-1} \cdots a_{m-r+1} / a_1 a_2 \cdots a_r \tag{7}$$

for m and r in Z^+ with $1 \le r \le m$.

If $a_n = n$, then $\begin{bmatrix} m \\ r \end{bmatrix}_+$ is the binomial coefficient $\binom{m}{r}$, which is well known to be an integer for m and r in Z^+ with $0 \le r \le m$. If a_n is the Fibonacci number F_n, then $\begin{bmatrix} m \\ r \end{bmatrix}$ is the Fibonomial coefficient $\left\{ \begin{matrix} m \\ r \end{matrix} \right\}$ given in (4); these coefficients are also known to be integers. In a later paper we shall show that conditions (5) and (6) imply that each generalized binomial coefficient $\begin{bmatrix} m \\ r \end{bmatrix}$ is an integer. Here we assume this result in our proof of a generalized Gould hexagon property.

Let p be a fixed positive prime. For all nonzero integers a let $E(a)$ denote the greatest integer e such that $p^e | a$.

In terms of this exponent function E, one can translate our hypotheses (5) and (6) into the two following statements:

$$\min \{E(a_r), E(a_s)\} \le E(a_{r+s}) \tag{8}$$

$$\min \{E(a_r), E(a_s)\} \le E(a_{|r-s|}) \ . \tag{9}$$

We now establish the following result:

Lemma 1. For all r and s in Z^+, no one of the integers

$$E(a_r), \qquad E(a_s), \qquad E(a_{r+s}) \tag{10}$$

is smaller than the other two integers in (10), i.e., the minimum integer in (10) occurs at least twice in (10).

Proof. If $E(a_{r+s})$ is a minimum of (10), we see from (8) that at least one of $E(a_r)$ and $E(a_s)$ does not exceed the minimum $E(a_{r+s})$ and hence is also a minimum in (10). If either $E(a_r)$ or $E(a_s)$ is a minimum in (10), then one can use (9) to show similarly that the minimum in (10) occurs at least twice in (10).

Using the definition (7) of the generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$, one can readily establish the proportionality relation

$$\begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix} : \begin{bmatrix} r+s-1 \\ r \end{bmatrix} : \begin{bmatrix} r+s \\ r \end{bmatrix} = a_r : a_s : a_{r+s} \ . \tag{11}$$

This proportion and Lemma 1 immediately give us

Lemma 2. The minimum integer in

$$E\left(\begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix}\right), \ E\left(\begin{bmatrix} r+s-1 \\ r \end{bmatrix}\right), \ E\left(\begin{bmatrix} r+s \\ r \end{bmatrix}\right) \tag{12}$$

occurs at least twice in (12). That is, if u, v, w are the terms of (12) in some order and $u < w$ then $u = v$.

Now let

$$C_0 = \begin{bmatrix} n \\ k \end{bmatrix}, \quad C_1 = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} n-1 \\ k \end{bmatrix}, \quad C_3 = \begin{bmatrix} n \\ k+1 \end{bmatrix},$$

(13)

$$C_4 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}, \quad C_5 = \begin{bmatrix} n+1 \\ k \end{bmatrix}, \quad C_6 = \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

The generalized Hoggatt-Hansell identity

(14) $$C_1C_3C_5 = C_2C_4C_6$$

is established in a straightforward manner. We now turn to the generalized Gould property

(15) $$\gcd(C_1, C_3, C_5) = \gcd(C_2, C_4, C_6) .$$

Let C_i be as in (13) and let $e_i = E(C_i)$ for $0 \leq i \leq 6$. Then (14) implies

$$e_1 + e_3 + e_5 = e_2 + e_4 + e_6 .$$

The Gould property (15) is equivalent to having (for all primes p)

(17) $$\min(e_1, e_3, e_5) = \min(e_2, e_4, e_6) .$$

If (17) were not true, then either

(18) $$e_i < \min(e_1, e_3, e_5) \qquad \text{for some } i \text{ in } \{2, 4, 6\}$$

or

(19) $$e_j < \min(e_2, e_4, e_6) \qquad \text{for some } j \text{ in } \{1, 3, 5\} .$$

We now assume the specific case

(20) $$e_1 < \min(e_2, e_4, e_6)$$

of (19) and show that (20) leads to a contradiction; the other cases of (18) and (19) lead to contradictions similarly.

The special case of (12) in which $r = k$ and $s = n - k$ is

(21) $$e_1, e_2, e_0 .$$

From (20) we have $e_1 < e_2$. This and Lemma 2 applied to (21) give us $e_1 = e_0$.

Now the inequality $e_1 < e_4$ from (20) and $e_1 = e_0$ tell us that $e_0 < e_4$. The case of (12) with $r = k + 1$ and $s = n - k$ is

(22) $$e_0,\ e_3,\ e_4\ .$$

Using $e_0 < e_4$ and applying Lemma 2 to (22) we find that $e_0 = e_3$.

The inequality $e_1 < e_6$ from (20) and $e_1 = e_0$ lead to $e_0 < e_6$. The case of (12) with $r = k$ and $s = n - k + 1$ is

(23) $$e_6,\ e_0,\ e_5\ .$$

Since $e_0 < e_6$, Lemma 2 applied to (23) gives us $e_0 = e_5$.

Thus we have shown that (20) implies

(24) $$e_0 = e_5 = e_3 = e_1 < \min(e_2, e_4, e_6)\ .$$

But it follows from (24) that $e_1 + e_3 + e_5 < e_2 + e_4 + e_6$ and this contradicts the consequence (16) of the Hoggatt-Hansell identity (14). Hence assumption (20) is false. Similarly, the other cases of (18) and (19) lead to contradictions. Therefore (17) is true and the generalized Gould property (15) is established.

It is now natural for people with Fibonacci interests to ask if properties (5) and (6) are true for sequences $\{a_n\}$ satisfying

(25) $$a_{n+2} = ca_{n+1} - da_n \qquad \text{for} \qquad n = 1, 2, 3, \cdots,$$

with c and d fixed integers. In a later paper, we shall show that if

(26) $$\gcd(c,d) = 1, \qquad a_1 = 1, \qquad \text{and} \qquad a_2 = c$$

then a sequence $\{a_n\}$ satisfying (25) has properties (5) and (6) and hence it gives rise to generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ that are integers with the Gould hexagon property (15).

If one drops the condition $\gcd(c,d) = 1$ in (26), then $\{a_n\}$ need not have properties (5) and (6) and the $\begin{bmatrix} m \\ r \end{bmatrix}$ need not have the Gould hexagon property. An example is the sequence

$$1,\ 2,\ 6,\ 16,\ 44,\ 120,\ 328,\ \cdots$$

with the recursion relation $a_{n+2} = 2a_{n+1} + 2a_n$. For this sequence, the $\begin{bmatrix} m \\ r \end{bmatrix}$ are all integers but the Gould hexagon property (15) is not true when $n = 5$ and $k = 2$.

REFERENCES

1. V. E. Hoggatt, Jr., and Walter Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2 (1971), pp. 120, 133.

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