# A SIMPLE OPTIMAL CONTROL SEQUENCE IN TERMS FIBONACCI NUMBERS

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## 1. INTRODUCTION

It is well known that the Fibonacci numbers are encountered in the optimization of the procedure for searching for the maximum or minimum value of a unimodal function [1-6]. The optimum search procedure can be derived by the method of dynamic programming [3, 4, 5, 6]. In the present note it is shown that the sequence of optimal control inputs, for a simple discrete-time system with a quadratic performance criterion, can be expressed in terms of the Fibonacci numbers.

## 2. A DISCRETE-TIME SYSTEM

Consider the very simple linear discrete-time system* described by the difference equation

(1) $$x(k+1) = x(k) + u(k) ,$$

where $u(k)$ is the control input to the system at discrete time instant $k$, and $x(k)$ is a state variable of the system at the same instant. Suppose that it is desired to find the sequence of $N$ control inputs $u(1), \cdots, u(N)$ which, starting from an initial system state $x(1)$, gives the minimum possible value to the summation $F$ defined by

(2) $$F = \sum_{k=1}^{N} [x^2(k) + u^2(k)] .$$

The final state $x(N)$ may be prescribed or not; assume for the present that the final state is zero.

This problem can easily be solved by dynamic programming [4-6]. The procedure is to start by supposing $N = 1$, use the solution of that simple problem to find the solution for $N = 2$, and proceed to derive a recurrence relationship which gives the solution of the problem for larger values of $N$. If we define the quantity $S_N(x)$ to be the minimum value of the summation $F$ reached in an N-stage process starting from the initial state $x$, we obtain the recurrence relationship

(3) $$S_N(x) = \min_u \{x^2 + u^2 + S_{N-1}(x+u)\}$$

*For a discussion of discrete-time systems, see, for example, [7].

The value of $S_1(x)$, for the specified endpoint $x(2) = 0$, can easily be seen to be

$$S_1(x) = 2x^2 \tag{4}$$

for the control input

$$u_1(1) = -x \quad , \tag{5}$$

where the notation $u_1(1)$ means the first (and only) input of the one-stage process, and where the initial state $x$ is understood. In this case there is really no optimization problem, as the specification of the final endpoint leaves no alternative but to choose $u = -x$ as given by (5). Having obtained the solution described by (4) and (5), however, we can proceed to find $S_2(x)$ by substitution in (3) as follows:

$$S_2(x) = \min_u \left\{x^2 + u^2 + 2(x + u)^2\right\} \tag{6}$$

Performing the minimization operation by differentiating the expression in braces with respect to $u$, we find that the optimum value of $u$ is given by

$$u_2(1) = -\frac{2}{3}x \quad , \tag{7}$$

where the notation $u_2(1)$ represents the first input of the two-stage process. Substituting (7) in (6), we obtain

$$S_2(x) = \frac{5}{3}x^2 \tag{8}$$

Based on Equations (4) and (8), suppose that

$$S_N(x) = K(N)x^2 \ . \tag{9}$$

$S_N(x)$ can be found by performing the minimizing operation involved in the expression

$$S_N(x) = \min_u \left\{x^2 + u^2 + K(N - 1)(x + u)^2\right\} \quad . \tag{10}$$

This minimization gives the value of $u$ to be

$$u_N(1) = \frac{-K(N - 1)}{K(N - 1) + 1}\, x \quad . \tag{11}$$

Substitution of (11) into (10) leads to the result

$$K(N) = \frac{2K(N - 1) + 1}{K(N - 1) + 1} \ . \tag{12}$$

We see from (12) that, if $K(N-1)$ is a rational number, $K(N)$ will also be rational. Therefore, because $K(2)$ is rational as shown by Eq. (8), $K(N)$ is rational for all values of $N$. If $K(N)$ is expressed in the form $A(N)/B(N)$, where $A$ and $B$ are integers with no common factor, the following results can be derived:

$$\frac{A(N)}{B(N)} = \frac{2A(N-1) + B(N-1)}{A(N-1) + B(N-1)} \tag{13}$$

$$A(N) = 2A(N-1) + B(N-1) \tag{14}$$

$$B(N) = A(N-1) + B(N-1). \tag{15}$$

The two first-order difference equations (14) and (15) can be expressed as a second-order difference equation (in either $A$ or $B$) of the form

$$A(N+1) - 3A(N) + A(N-1) = 0 \quad . \tag{16}$$

Compare Eq. (16) with the following equation for the Fibonacci numbers $F(n)$ for values of $n$ separated by two units instead of one:

$$F(n+2) - 3F(n) + F(n-2) = 0 \; . \tag{17}$$

Equation (17) can easily be obtained from the basic equation for the Fibonacci numbers

$$F(k) = F(k-1) + F(k-2) \tag{18}$$

by taking $k = n,\ n+1,\ n+2,$ and manipulating the three equations so obtained. Comparing Eqs. (16) and (17), and using the initial conditions given by Eq. (8), it is found that $K(N)$ can be expressed in the form

$$K(N) = \frac{F(2N+1)}{F(2N)} \quad , \tag{19}$$

where $F(k)$ is the Fibonacci number defined by (18) with initial conditions $F(0) = 0$, $F(1) = 1$. Equation (11) leads to the result

$$u_N(1) = \frac{-F(2N-1)}{F(2N)}\, x \quad . \tag{20}$$

This result shows that the optimal control input is a function of the present state and the number of stages to go to the end of the process.

If the input given by Eq. (20) is applied to the system in initial state $x(1)$, the next state $x(2)$ is given by

$$x(2) = \left[1 - \frac{F(2N-1)}{F(2N)}\right] x(1) = \frac{F(2N-2)}{F(2N)}\, x(1) \quad . \tag{21}$$

The next input $u_N(2)$ can be expressed in the form

$$u_N(2) = \frac{-F(2N-3)}{F(2N-2)} \frac{F(2N-2)}{F(2N)}\, x(1) = \frac{-F(2N-3)}{F(2N)}\, x(1) \quad . \tag{22}$$

The sequence of optimal control inputs $u_N(i)$ can therefore be expressed in the form

$$u_N(i) = \frac{-F(2N-2i+1)}{F(2N)}\, x(1) \qquad (i = 1, \cdots, N) \ . \tag{23}$$

If the final state is unspecified and therefore allowed to take on any value, the value of the last control input $u_N(N)$ is zero, and the values of $K(N)$ and $u_N(1)$ can be expressed in the forms

$$K(N) = \frac{F(2N)}{F(2N-1)} \tag{24}$$

$$u_N(1) = \frac{-F(2N-2)}{F(2N-1)}\, x \quad . \tag{25}$$

The optimal sequence of control inputs $u_N(i)$ is in this case given by

$$u_N(i) = \frac{-F(2N-2i)}{F(2N-1)}\, x(1) \qquad (i = 1, \cdots, N) \quad . \tag{26}$$

These results are discussed more fully, and compared with the optimal control input for a continuous-time system, in [6].

## REFERENCES

1. R. Bellman, Dynamic Programming, Princeton University Press, 1957, pp. 34-36.
2. L. T. Oliver and D. J. Wilde, "Symmetric Sequential Minimax Search for a Maximum," Fibonacci Quarterly, Vol. 2, No. 3, Oct., 1964, pp. 169-175.
3. D. J. Wilde, Optimum Seeking Methods, Prentice-Hall, 1964.

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