FIBONACCI DRAINAGE PATTERNS

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1. INTRODUCTION

Some 25 years ago, an engineer, Robert E. Horton, developed the notion of stream order or class [1] as a measure of the position of a stream in the hierarchy of tributaries as observed in natural river basins. Using a map of a specified scale, he designated first-class streams as those which have no tributaries, second-class streams as those which have tributaries only of the first class, and third-class streams as those which have only first- and second-class tributaries, and so on (Fig. 1). Although Horton in his original analysis next renumbered the streams to show the headward extent of the main thread of the river, it has since been shown [6] that for most purposes this is an unnecessary complication. If the renumbering procedure is omitted then the basic property of stream class numbers is that if two streams of the same class i combine [5], the resulting stream is of class i + 1, that is

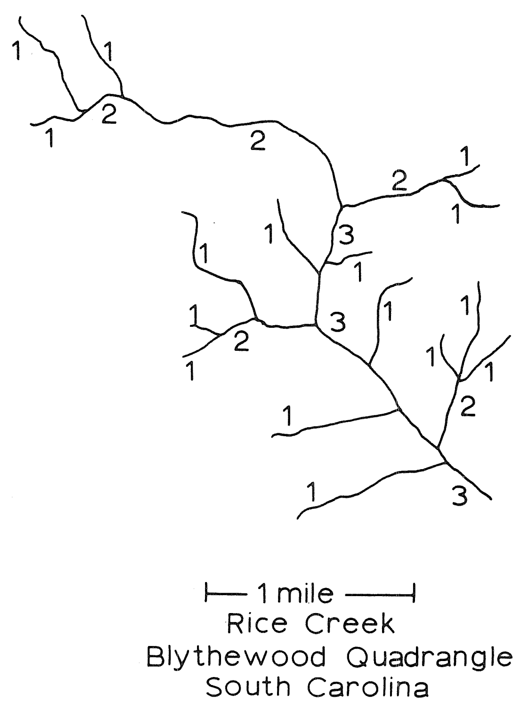

Fig. 1 A Third-Class Natural Stream Illustrating the Ordering Procedure

$$i * i = i + 1,$$

where an asterisk (*) signifies the junction or combination of two streams. If, however, the streams are of different class, then the lower class stream is lost in the one of higher class and the combination is expressed by [5]:

$$i * j = j * i = j \qquad (j > i).$$

Using this system of stream classification, Horton [1] noticed that in natural river basins when the logarithm of the number of streams of each class was plotted versus the stream class (Fig. 2), the graph formed a straight line. The constant slope of the line, Horton called the bifurcation ratio. Measured values of this ratio for natural streams range between 2 and 5 but seem to average about 3.5 [2, p. 138]. Although many geomorphologists have been greatly intrigued by this natural relationship between numbers of streams and their class, it is still not clearly understood why the relationship holds so well.

Because the bifurcation ratio is given by

$$r_b = \frac{N_i}{N_i + 1}$$

Horton then summarized the result of his observations for a basin using the relation

$$N_i = r_b^{m-i},$$

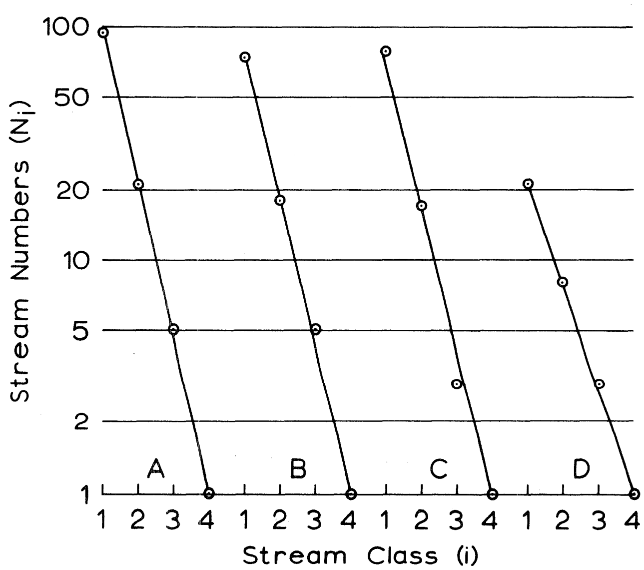

Fig. 2 A Graph Showing the Relationships Between the Stream Class and the Number of Streams in each Class. Examples are: (A) Hightower Creek, Georgia [9, p. 19]; (B) Tar Hollow, Ohio [4, p. 1036]; (C) Green Lick, Pa. [4, p. 1036]; (D) Fibonacci Pattern (8th Order)

where m is the class of the main stream in the basin and N_i is the number of streams of each class i. This equation is analogous to the simple population growth equation [3, p. 129] when the initial population is taken equal to one. The bifurcation ratio then corresponds to the net reproductive rate, m - i corresponds to the number of generations and N_i corresponds to the population size after m - i generations (Fig. 3).

Norton's equation by itself implies that drainage nets only have junctions of the type where streams of class i meet at a single place to form streams of class i + 1 (Figs. 3 and 4). That is, only junctions of the type

$$i * i = i + 1$$
$$i * i * i = i + 1$$
$$i * i * i * i = i + 1$$

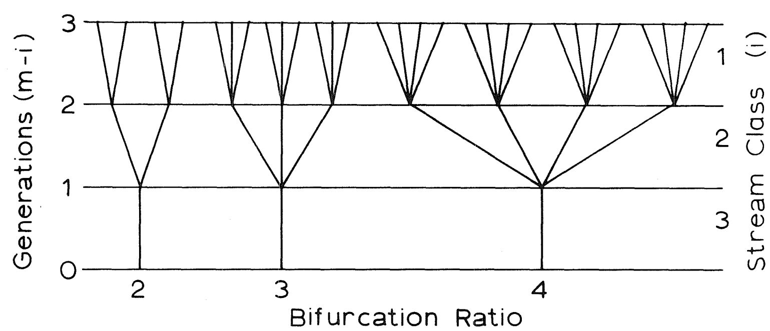

Fig. 3. Branching Systems for Various Bifurcation Ratios which Obey the Simple Population Growth Equation

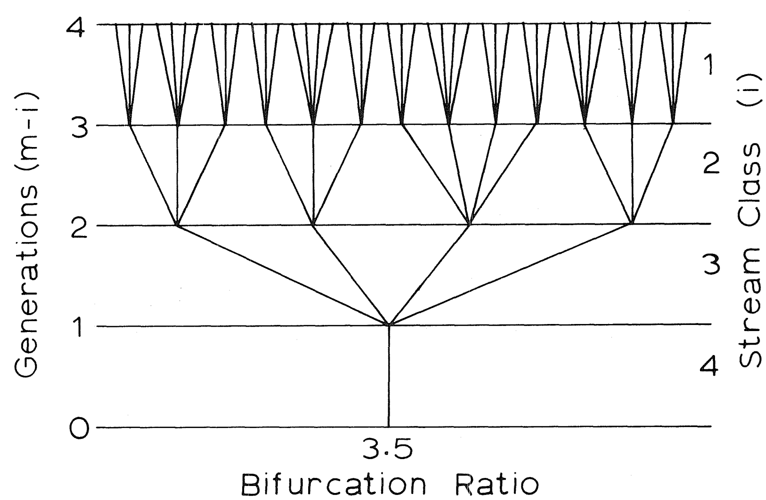

Fig. 4. A Simple Branching System having an Average Bifurcation Ratio of 3.5

etc., exist with no junctions of the type

$$i * j = j \qquad (j > i) .$$

This would suggest that natural streams should appear as shown in Fig. 4. However, this branching pattern has little resemblance to natural drainage patterns (Fig. 1) because there are no junctions of the type where a low-class stream becomes lost in a higher-class stream.

If we next examine the drainage patterns of randomly selected small second- and third-class streams, one will soon notice that a significant percentage of these patterns resemble the branching pattern obtained by constructing a Fibonacci tree [10, p. 47]. This resemblance can be illustrated by comparing an unnamed portion of Rice Creek and the upper reaches of Crane Creek (Figs. 5 and 6) Blythewood Quadrangle, South Carolina with the 5^{th} and 6^{th} order Fibonacci trees (Fig. 7), respectively.

2. SIMPLE PROPERTIES OF FIBONACCI DRAINAGE PATTERNS

If a Fibonacci tree of any order is treated as if it were a drainage pattern [7], we can apply Horton's numbering procedure to the branches of the tree and call these branches streams (Fig. 7). Inspection of any such Fibonacci tree (Fig. 7) shows that the total number

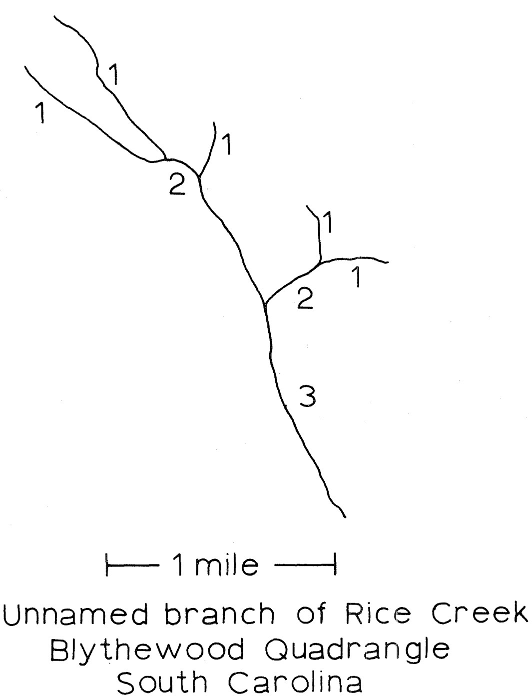

Fig. 5. A Third-Class Natural Stream Which can be Represented as a Fifth-Order Fibonacci Tree

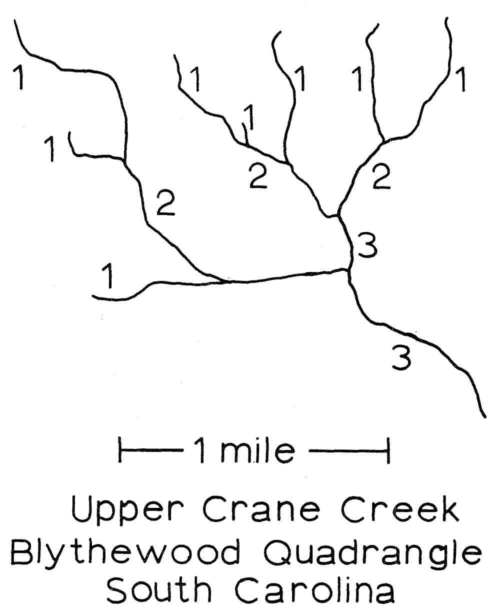

Fig. 6. A Third-Class Natural Stream Which can be Represented as a Sixth-Order Fibonacci Tree

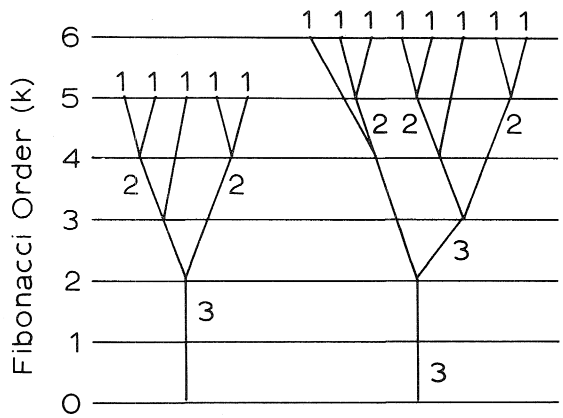

Fig. 7. Fibonacci Trees of Fifth and Sixth Order

of pendant vertices (first-class streams) at the end of k Fibonacci orders is given by F_k, the Fibonacci number having index k where F_k is evaluated using the recursion formula:

$$F_k = F_{k-1} + F_{k-2}$$

and the initial conditions:

$$F_{-k} = 0, \qquad F_0 = 0, \qquad F_1 = 1$$

so that in a Fibonacci stream basin

$$N_1 = F_k .$$

Similarly, inspection of the Fibonacci tree (Fig. 7) having a maximum Fibonacci order of k shows that the total number of streams of class i is given by $N_i = F_n$ where $n = k - 2(i - 1)$ and the total number of streams of i^{th} class lost in a stream of class $i + 1$ is given by $L_i = F_\ell$ where $\ell = k - 2i - 1$. The stream of highest class (m) in a tree of Fibonacci order k is given by

$$m = \left[\frac{k+1}{2}\right]$$

where [ ] signifies the integral value. The bifurcation ratio is given by

$$\frac{N_i}{N_{i+1}} = \frac{F_{k-2i+2}}{F_{i-2i}} = 1 + \frac{F_{k-2i+1}}{F_{k-2i}}$$

and for large values of $(k - 2i)$, this ratio converges to the constant $1 + \tau$ where τ is the famous golden ratio. Comparison of the Fibonacci bifurcation ratio of 2.618 (Fig. 2) with the average ratio of 3.5 shows the Fibonacci ratio to be significantly smaller than the ratio seen in natural streams.

In Fibonacci streams, the disappearance of low-class streams into higher-class streams has the following fixed restrictions:

1. When $i * j = j$, then $0 \le j - 1 \le 1$.
2. No more than one stream of class i can be lost in any given stream of class $i + 1$.

3. A GENERALIZED FIBONACCI DRAINAGE PATTERN

The restriction imposed on Fibonacci patterns that no more than one i^{th} class stream can be lost in any given $i + 1$ class stream can be partially relaxed by considering a simple form of a generalized Fibonacci tree [8, p. 922]. In the usual construction of a Fibonacci tree [10, p. 47] it is assumed that the trunk and each limb has a maturing time of one period and a gestation time of one period. This growth pattern can be modified by changing the maturing period from one to some other integral period (Fig. 10).

If p is equal to one (gestation period) plus the maturing period (any integer greater than one) and k is the total number of elapsed periods (order), then ${}^pF_k$ is a generalized Fibonacci number which can be evaluated from the recursion formula:

$$ {}^pF_k = {}^pF_{k-1} + {}^pF_{k-p} $$

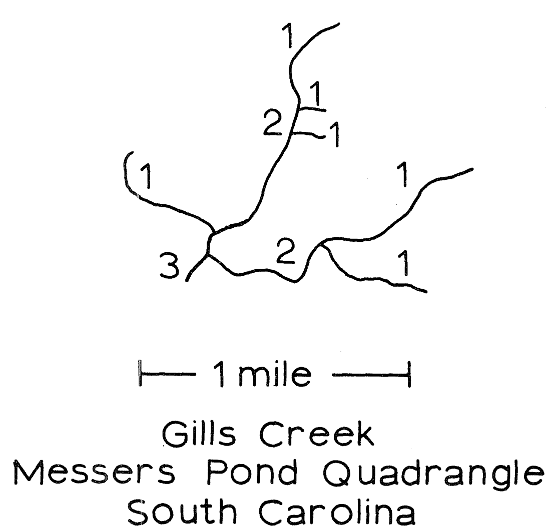

Fig. 8. A Third-Class Natural Stream Which Can Be Represented as a Seventh-Order Modified Fibonacci Tree with Two-Period Maturation

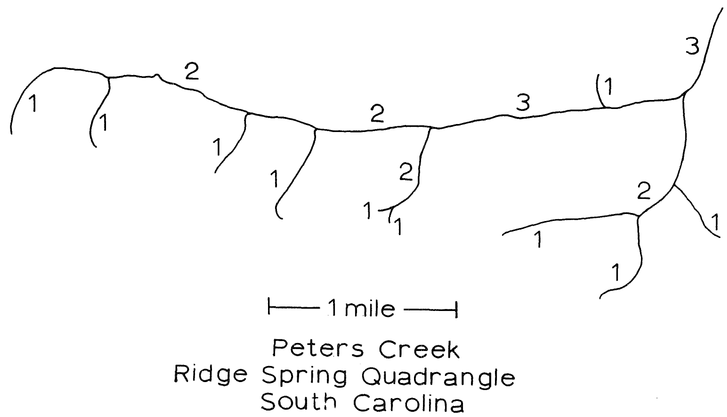

Fig. 9. A Third-Class Natural Stream Which can be Represented as an Eighth-Order Modified Fibonacci Tree with Two-Period Maturation

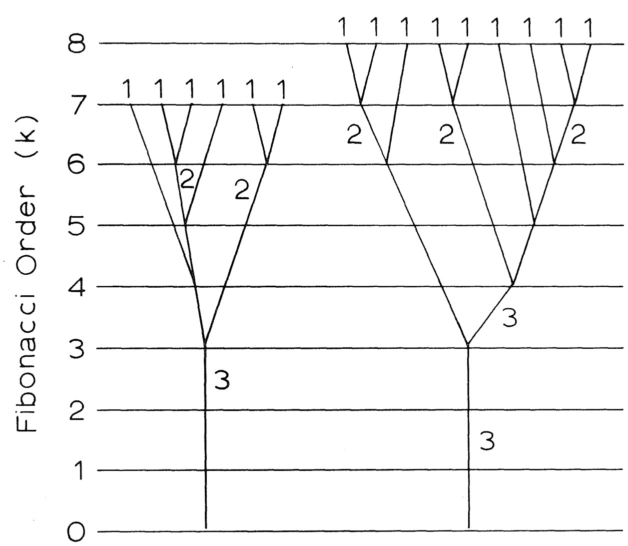

Fig. 10. Modified Fibonacci Trees of Seventh and Eighth Order with Two-Period Maturation

and the initial conditions:

$$p_{F_{-k}} = 0, \qquad p_{F_0} = 0, \qquad p_{F_1} = 1 .$$

In a manner similar to regular Fibonacci patterns this group of generalized Fibonacci patterns has a total number of pendant vertices (first-class streams) at the end of k orders given by

$$N_1 = {}^pF_k .$$

The total number of streams of class i is then given by

$$N_i = {}^pF_n ,$$

where

$$n = k - p(i - 1)$$

the total number of streams i lost in class i + 1 is given by

$$L_i = {}^pF_\ell - {}^pF_{k-pi} ,$$

where

$$\ell = k - 1 - p(i - 1) ,$$

and the bifurcation ratio is given by

$$r_b = \frac{N_i}{N_{i+1}} = \frac{{}^pF_{k-p(i-1)}}{{}^pF_{k-pi}} = 1 + \frac{{}^pF_{k-1-p(i+1)}}{{}^pF_{k-pi}} ,$$

which will converge to a constant for large values of $(k - pi)$.

This group of generalized Fibonacci streams has the following fixed restrictions governing the loss of low-class streams into higher-class streams:

1. when $i * j = j$, then $0 \le j - i \le 1$,
2. No more than $(p - 1)$ streams of class i can be lost in any given stream of class $i + 1$.

Natural streams which resemble this type of generalized Fibonacci pattern are illustrated by comparing the upper reaches of Gills Creek (Fig. 8) Messers Pond Quadrangle, and Peters Creek (Fig. 9) Ridge Spring Quadrangle, South Carolina with 7th and 8th order generalized Fibonacci trees (Fig. 10) having two-period maturation $(p = 3)$.

4. CONCLUSION

Because natural second-class streams can only have first-class tributaries and third-class streams can only have first- and second-class tributaries, the very restrictive junction rule $(|j - i| \le 1)$ for Fibonacci patterns is commonly satisfied. This produces a superficial resemblance between these small natural patterns and Fibonacci patterns. In fourth- and higher-class basins the opportunities for violation of the Fibonacci junction rule are suddenly increased and the resemblance rapidly fades. Yet even in basins of the highest class, whenever the branching among two or three adjacent classes is emphasized, a Fibonacci pattern can often be discerned.

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