ON SUMMATIONS AND EXPANSIONS OF FIBONACCI NUMBERS

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INTRODUCTION

One of the early delights a neophyte in the study of Fibonacci numbers experiences may be an encounter with some elementary summation properties such as

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 .$$

As soon as his curiosity is aroused, he may wish to investigate summations which "skip" a constant number of Fibonacci numbers, for instance the problem of obtaining a formula for the sum of the first n Fibonacci numbers of odd position index.

But — as has often been observed — mathematicians are like lovers: give them the little finger, and they will want the whole hand. Can one find a relationship which spells out the sum of any finite Fibonacci sequence whose subindices follow the pattern of an arithmetic progression?

A SUMMATION THEOREM (Theorem 1)

Seeking a pattern for the sum of a number of equally spaced Fibonacci numbers means a concern with

$$\sum_{i=0}^{n} F_{n_i} , \qquad (n_i = k\,i+r) ,$$

r is a non-negative integer, whereas k is a natural number.

Let us use the Binet formula

$$F_n = \frac{a^n - b^n}{\sqrt{5}} \quad \text{with} \quad a = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{5}}{2} .$$

We also note that $ab = -1$. The n^{th} Lucas number, L_n, is $L_n = a^n + b^n$. Then

$$\sum_{i=0}^{n} F_{n_i}$$

becomes:

$$\frac{1}{\sqrt{5}}\sum_{i=0}^{n}(a^{ik+r} - b^{ik+r}) = \frac{1}{\sqrt{5}}\left[a^r\,\frac{a^{(n+1)k}-1}{a^k-1} - b^r\,\frac{b^{(n+1)k}-1}{b^k-1}\right]$$

$$= \frac{[a^{(n+1)k+r} - a^r](b^k-1) - [b^{(n+1)k+r} - b^r](a^k-1)}{\sqrt{5}\,[(-1)^k + 1 - L_k]} .$$

Performing the indicated operations and again employing the Binet formula, we are ready to give the sum of n Fibonacci numbers beginning with F_r. The sequence continues equally spaced such that $(k-1)$ Fibonacci numbers are left out from any one term to the next.

Theorem 1.

$$\sum_{i=1}^{n} F_{k(i-1)+r} = \frac{(-1)^k F_{(n-1)k+r} + (-1)^{\min(k,r)+t} F_{|r-k|} - F_{nk+r} + F_r}{(-1)^k + 1 - L_k} ,$$

where k is any natural number and r any non-negative integer. The number t is defined by:

$$t = \begin{cases} 0, & \text{when } r < k \\ 1, & \text{when } r > k \end{cases} .$$

Since $F_{|r-k|}$ vanishes for $r = k$, t need not be defined in this case.

Attention should be drawn to the fact that we may restrict r to the condition $0 \le r < k$ by the

Reduction Formula: (2)

If $r \equiv \bar{r} \pmod{k}$, i.e.: $r = ak + \bar{r}$ where a is a natural number and $0 \le \bar{r} < k$, then

$$\sum_{i=1}^{n} F_{(i-1)k+r} = \sum_{i=1}^{n} F_{(a+i-1)k+\bar{r}}$$

$$= \sum_{i=1}^{n+a} F_{(i-1)k+\bar{r}} - \sum_{i=1}^{a} F_{(i-1)k+\bar{r}} .$$

While the restriction on $\bar{r}$ is useful for reduction purposes, it is not a necessary condition for relationship (2).

Special Cases of Theorem 1.

We notice that the result of our summation involves an expression which combines no more than four terms. Thus, this relationship would be quite helpful whenever n is "fairly large." For $r = 0$, the special case

(3)
$$\sum_{i=1}^{n} F_{ki} = \frac{(-1)^k F_{nk} + F_k - F_{(n+1)k}}{(-1)^k + 1 - L_k}$$

may merit attention.

It is evident that Theorem 1 embraces the basic elementary summation formulas of this kind. Obviously, $k = 1$, $r = 0$ yields:

$$\sum_{i=1}^{n} F_i = F_n + F_{n+1} - 1 = F_{n+2} - 1 ,$$

which is the formula we previously quoted for the sum of the first n Fibonacci numbers.

However, it is aesthetically satisfying that the summation formulas for the first n Fibonacci numbers of odd indices and those of even indices also become special cases of our pattern. Thus, by letting $k = 2$ and $r = 0$, we get

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1 ,$$

whereas $r = 1$ yields:

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n} .$$

If one relationship combining the two cases were required, Theorem 1 — for $k = 2$ and $r = 0$ or 1 — becomes:

$$\sum_{i=1}^{n} F_{2(i-1)+r} = F_{2n+2-2} - (-1)^r F_{2-r} - F_r ,$$

or, more simply:

(4)
$$\sum_{i=1}^{n} F_{2(i-1)+r} = F_{2n+r-1} + r - 1 .$$

It may be instructive to check other cases of small "skipping numbers" k. Owing to reduction formula (2), the condition $r < k$ does not limit the generality of the results.

For $k = 3$ we obtain

$$\sum_{i=1}^{n} F_{3(i-1)+r} = \frac{2F_{3n+r-1} - (-1)^r F_{3-r} - F_r}{4} ,$$

which may also be stated as

(5) $$\sum_{i=1}^{n} F_{3(i-1)+r} = \frac{F_{3n+r-1} - |r - 1|}{2} .$$

and, for $k = 4$, we have

$$\sum_{i=1}^{n} F_{4(i-1)+r} = \frac{2F_{4n+r-3} + F_{4n+r-2} - (-1)^r F_{4-r} - F_r}{5}$$

or, alternatively:

(6) $$\sum_{i=1}^{n} F_{4(i-1)+r} = F_{2n-2}F_{2n+r} + \left[\frac{r+1}{2}\right] .$$

These equivalences, relationships (5) and (6), may easily be verified by straight substitution of the few r-values to which we are restricted. All of these formulas can, however, readily be established either by using the Binet formula, or else, employing mathematical induction. They were stated here merely as a matter of interest since none of them seem too obvious.

Two further observations may be mentioned.

We might wish to impose the condition $r = k$ on Theorem 1. Then

(7) $$\sum_{i=1}^{n} F_{ik} = \frac{(-1)^k F_{nk} - F_{(n+1)k} + F_k}{(-1)^k + 1 - L_k} .$$

Clearly, the summation formula for the first n Fibonacci numbers of even subindex is a special case of this.

It may also be of interest to note that on the basis of Theorem 1, L_k divides into all sums of our kind, provided k is odd, i.e., the number of Fibonacci numbers "skipped over" in our summation is even. If this number were odd, $(2 - L_k)$ would be a divisor of our sum.

AN EXPANSION THEOREM (Theorem 2)

But hasn't Jacobi advised us: "Man muss immer umkehren" (one must always turn around)? Thus — having obtained summation results as expressions involving Fibonacci numbers — we may now experiment with an inversion and pose the problem: **Can a Fibonacci** number be expanded into a series reminiscent of an expansion for the n^{th} power of a binomial?

Partly analogous to Theorem 1, and primarily for the sake of developing some notions, we symbolize our Fibonacci numbers F_n as F_{km+r}, where all letters represent non-negative integers.

The proposed expansion reads:

Theorem 2.

$$F_n = \sum_{i=0}^{k-1} \binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i F_{m+r-k+i+1}, \qquad (n = km + r).$$

In our proof, we use mathematical induction on n. Symbolizing Theorem 2 by R(n), we readily verify R(n) for the first few natural numbers. Now we need to show that the correctness of R(s - 1) and of R(s) implies correctness of R(s + 1), where s represents any natural number. This means that we investigate whether

$$\binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i [F_{m+r-k+i} + F_{m+r-k+i+1}]$$

equals

$$\binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i F_{m+r-k+i+2}.$$

However, the iterative definition of Fibonacci numbers assures the correctness of this equality and, hence, completes the proof.

As an illustration, we might wish to expand F_{11} by letting m = 3 and r = 2. We assert that

$$F_{11} = \sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{3+i},$$

which is easily verified.

Special Cases of Theorem 2.

Some special cases might be pointed to. Considering Fibonacci numbers with even subindex, Theorem 2 reduces to:

$$F_n = \sum_{i=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{i} 2^i F_{3-(n/2)+i}. \tag{8}$$

But those of odd subindex may be expanded on the basis of

$$F_n = \sum_{i=0}^{\frac{n-3}{2}} \binom{\frac{n-3}{2}}{i} 2^i F_{(9-n)/2+i}. \tag{9}$$

A Corollary of Theorem 2.

We propose a corollary of expansion formula 2 (Theorem 2) which gives a prescribed number of terms for the expansion. Let the symbol a stand for that number. In our condition $n = km + r$ we stipulate that $m = 1$ and $k = a$, and we obtain:

Corollary of Theorem 2.

$$F_n = \sum_{i=0}^{a-1} \binom{a-1}{i} F_{n+2(1-a)+i} , \tag{10}$$

where

$$2 \le a \le \frac{n+1}{2} .$$

Special Cases of the Corollary:

The following two special cases seem worth mentioning. We desire to let a be the largest possible number.

Case 1:

If n is even, $a = n/2$ is chosen. Then

$$F_n = \sum_{i=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{i} F_{i+2} \tag{11}$$

and there are $n/2$ terms in the expansion.

Case 2:

If n is odd, $a = \frac{n+1}{2}$,

$$F_n = \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} F_{i+1} \tag{12}$$

and the expansion has $\frac{n+1}{2}$ terms.

To illustrate, let us expand F_{21} into a five-term series. Then $n = 21$. Using relationship (10) and letting $a = 5$, we have:

$$F_{21} = \sum_{i=0}^{4} \binom{4}{i} F_{13+i} ,$$

which is correct. For the maximum number of terms in the expansion we would designate a as being 11 and use (12). Then

$$F_{21} = \sum_{i=0}^{10} \binom{10}{i} F_{i+1} ,$$

a relationship which can also be easily verified.

BACK TO ANOTHER SUMMATION THEOREM (Theorem 3)

Once again, we might "invert." Our summation theorem (Theorem 1) gave us an expansion involving Fibonacci numbers as the result of the addition. Now let us give a summation which results in one Fibonacci number. This problem may possibly use Theorem 2 to the best advantage.

Starting with a summation involving Fibonacci numbers of prescribed indices, can we predict the resulting Fibonacci number? Again recalling Jacobi's advice, we reverse the expansion of a given Fibonacci number to a sum. Now designate a sum which leads to a predictable Fibonacci number. Symbolize m by u, and $u + r - (n - r)/u + 1$ by v. Then $r = v - 1 - u + k$ and Theorem 2 becomes:

Theorem 3.

$$\sum_{i=0}^{k-1} \binom{k-1}{i} F_u^{k-1-i} F_{u+1}^i F_{v+i} = F_{(k-1)(u+1)+v}$$

for any arbitrarily chosen natural numbers u and v. The number k may be any integer greater than or equal to 2.

To illustrate this summation idea, we try a summation involving F_4 and F_7. Here we let $u = 4$, and $v = 7$, and get:

$$\sum_{i=0}^{k-1} \binom{k-1}{i} 3^{k-1-i} 5^i F_{7+i} .$$

We predict F_{5k+2} as our result which is correct.

Pre-assigning the Fibonacci Number Resulting from Summation Theorem 3:

Formula 3 is a method for a summation which uses prescribed Fibonacci numbers and predicts a Fibonacci number as the result. What about assigning the resulting Fibonacci number without prescribing Fibonacci numbers involved in the summation?

This summation, not necessarily unique, can be had by considering two cases.

Case 1. The Fibonacci number to be attained has odd subindex n. We choose $u = v = 1$, and have

$$\sum_{i=0}^{k-1} \binom{k-1}{i} F_{i+2} = F_{2k-1} . \tag{13}$$

Case 2. We wish to obtain a Fibonacci number of even subindex. For this purpose we let u and v take on the values 1 and 2, respectively. Here:

$$\sum_{i=0}^{k-2} \binom{k-1}{i} F_{i+2} = F_{2k} . \tag{14}$$

Obviously, the number of terms in these summations will be $(n+1)/2$ for odd subindices n, and $n/2$ for even ones. We realize, however, that our choices for u and v have forfeited the ability to discern the powers of F_u and F_v which characterize the terms of Theorem 3.

Pre-Assigning the Fibonacci Number Resulting from Summation Theorem 3 as well as the Number of Terms in the Summation, and Retaining Generality.

Finally, we prescribe the resulting Fibonacci number F_n as well as k, the number of terms in the summation. Moreover, to avoid the difficulty encountered above, exclude the somewhat trivial cases which involve $F_1 = F_2 = 1$ among the summation terms. We impose the condition: $u, v \geq 3$. Furthermore, the iterative definition of Fibonacci numbers:

$$\sum_{i=0}^{1} F_{n+i} = F_{n+2}$$

inherently provides a summation of two terms resulting in a Fibonacci number (even though the summation is not of our general type). Therefore, impose the condition: $k \geq 3$. Then, for all $n \geq 4k-1$; i.e., for all $n \geq 11$, we can do justice to our data by assigning appropriate values to u and v such that

$$n = (k-1)(u+1) + v \tag{15}$$

is satisfied. Again, no claim is made for uniqueness.

For example, to obtain F_{11} through a summation of three terms, the following choice proves successful:

$$\sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{3+i} = F_{11} .$$

For a summation of three terms for F_{15}, we can already write:

$$\sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{7+i} = \sum_{i=0}^{2} \binom{2}{i} F_4^{2-i} F_5^i F_{5+i} = \sum_{i=0}^{2} \binom{2}{i} F_5^{2-i} F_6^i F_{2+i} = F_{15} .$$

Lack of Uniqueness — Predicting the Number of Different Summations

Can you foretell the number of different summation representations of our type, each having k terms, and leading to the same Fibonacci number F_n? Using relationship (15), our prediction becomes:

If set T is defined by

$$T = \left\{ t : 4 \le t \le \frac{n-3}{k-1} \right\},$$

then the numerosity of T, that is, the number

$$\left[\frac{n-3}{k-1}\right] - 3 \tag{16}$$

predicts the possible number of different summations of our type, each having k terms and leading to the Fibonacci number F_n.

To illustrate, there will be 52 ten-term summations of our kind leading to F_{500}. We would have:

$$\sum_{i=0}^{9} \binom{9}{i} F_{54}^{9-i} F_{55}^{i} F_{5+i} = \sum_{i=0}^{9} \binom{9}{i} F_{53}^{9-i} F_{54}^{i} F_{14+i} = \sum_{i=0}^{9} \binom{9}{i} F_{52}^{9-i} F_{53}^{i} F_{23+i}$$

$$= \cdots = \sum_{i=0}^{9} \binom{9}{i} F_{3}^{9-i} F_{4}^{i} F_{464+i} = F_{500} \,.$$

[Continued from page 62.]

then $V_n = L_n$, the Lucas sequence, and so (III) now gives the correct expression for (9) in (*).

Case 2. $A + B = 0$. We now obtain from (II)

$$\frac{f(x + c_1) - f(x + c_2)}{c_1 - c_2} = \sum_{n=0}^{\infty} \frac{U_n}{n!} D^n f(x) , \tag{IV}$$

where $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = PU_{n+1} - QU_n$. Thus for $P = 1$, $Q = -1$, $U_n = F_n$; and for $P = 2$, $Q = -1$, $U_n = P_n$, the Pell sequence. For $m = 1, 2, \cdots$, we obtain from (IV)

$$\frac{f(x + c_1^m) - f(x + c_2^m)}{c_1 - c_2} = \sum_{n=0}^{\infty} \frac{V_{mn}}{n!} D^n f(x) \,. \tag{V}$$

Remarks. The same ideas in (*) show that the generating function of the moments of the inverse operator

[Continued on page 84.]