ON THE GREATEST COMMON DIVISOR OF SOME BINOMIAL COEFFICIENTS

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Henry W. Gould [1] has raised the conjecture

$$(1) \qquad \gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\},$$

which we shall prove in this note.

It is convenient to express the proof in terms of the p-adic valuation of rationals.

<u>Definition.</u> Let $r = p^{\alpha}(a/b)$ where $(a,p) = (b,p) = 1$ then $|r|_p = p^{-\alpha}$.

We need only two properties of this valuation.

$$(2) \quad \text{Ultrametric inequality.} \qquad |a + b|_p \le \max\{|a|_p, |b|_p\} \quad ;$$

and for all integera $a_1, \cdots, a_n$ we have

$$(3) \qquad |\gcd(a_1, \cdots, a_n)|_p = \max\{|a_1|_p, \cdots, |a_n|_p\} .$$

In view of (3) we can rephrase (1) as follows.

<u>Conjecture.</u> For all primes p we have

$$(4) \quad \max\left\{\left|\binom{n-1}{k}\right|_p, \left|\binom{n}{k-1}\right|_p, \left|\binom{n+1}{k+1}\right|_p\right\} = \max\left\{\left|\binom{n-1}{k-1}\right|_p, \left|\binom{n}{k+1}\right|_p, \left|\binom{n+1}{k}\right|_p\right\}.$$

If we divide both sides by

$$\left|\frac{(n-1)(n-2)\cdots(n-k+2)}{(k+1)!}\right|_p$$

we get the equivalent conjecture

$$(5) \quad \begin{aligned} M_1(n,k) &= \max\{|(n-k)(n-k+1)(k+1)|_p, |nk(k+1)|_p, |(n+1)n(n-k+1)|_p\} \\ &= \max\{|(n-k+1)k(k+1)|_p, |n(n-k)(n-k+1)|_p, |(n+1)n(k+1)|_p\} \\ &= M_2(n,k) \quad . \end{aligned}$$

It thus suffices to prove $M_1 \le M_2$ and $M_2 \le M_1$ by deriving contradictions from the assumptions that one of the terms in M_1 exceeds M_2 or one of the terms in M_2 exceeds M_1. Since $M_2(n,k) = M_1(-k-1, -n-1)$ this involves only three steps.

Step 1. If

$$|(n - k)(n - k + 1)(k + 1)|_p > M_2$$

then

$$|k|_p < |n - k|_p \le 1, \quad \text{so} \quad |k + 1|_p = 1$$

$$|n|_p < |k + 1|_p = 1, \quad \text{so} \quad |n + 1|_p = 1$$

$$|n|_p = |n(n + 1)|_p < |n - k|_p \le \max\{|n|_p, |k|_p\} < |n - k|_p$$

a contradiction.

Step 2. If

$$|nk(k + 1)|_p > M_2$$

then

$$|n - k + 1|_p < |n|_p \le 1 \quad \text{so} \quad |n - k|_p = 1$$

$$|n - k + 1|_p = |(n - k)(n - k + 1)|_p < |k(k + 1)|_p \le |k|_p$$

$$|n + 1|_p < |k|_p = |(n + 1) - (n - k + 1)|_p \le \max\{|n + 1|_p, |n - k + 1|_p\}$$
$$< |k|_p$$

a contradiction.

Step 3. If

$$|(n + 1)n(n - k + 1)|_p > M_2$$

then

$$|k(k + 1)|_p < |n(n + 1)|_p \le |n + 1|_p$$

$$|n - k|_p < |n + 1|_p$$

$$|k + 1|_p < |n - k + 1|_p \quad \text{so} \quad |k|_p = 1 .$$

The first inequality now yields

$$|k + 1|_p < |n + 1|_p = |(n - k) + (k + 1)|_p \le \max\{|n - k|_p, |k + 1|_p\}$$
$$< |n + 1|_p$$

a contradiction.

We have thus completed the proof of $M_1(n,k) \le M_2(n,k) = M_1(-k - 1, -n - 1)$ and hence by symmetry the proof of $M_2(n,k) = M_1(-k - 1, -n - 1) \le M_2(-k - 1, -n - 1) = M_1(n,k)$.

REFERENCE

1. H. W. Gould, "A New Greatest Common Divisor Property of Binomial Coefficients," Abstract* 72T-A248, Notices AMS 19 (1972), A685.

See the December issue for two pertinent articles.