UNIT DETERMINANTS IN GENERALIZED PASCAL TRIANGLES

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There are many ways in which one can select a square array from Pascal's triangle which will have a determinant of value one. What is surprising is that two classes of generalized Pascal triangles which arose in [1] also have this property: the multinomial coefficient triangles and the convolution triangles formed from sequences which are found as the sums of elements appearing on rising diagonals within the binomial and multinomial coefficient triangles. The generalized Pascal triangles also share sequences of $k \times k$ determinants whose values are successive binomial coefficients in the k^{th} column of Pascal's triangle.

1. UNIT DETERMINANTS WITHIN PASCAL'S TRIANGLE

When Pascal's triangle is imbedded in matrices throughout this paper, we will number the rows and columns in the usual matrix notation, with the leftmost column the first column. If we refer to Pascal's triangle itself, then the leftmost column is the zeroth column, and the top row is the zeroth row.

First we write Pascal's triangle in rectangular form as the $n \times n$ matrix $P = (p_{ij})$.

$$(1.1) \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ 1 & 5 & 15 & 35 & 70 & 126 & \cdots \\ 1 & 6 & 21 & 56 & 126 & 252 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n} ,$$

where the element p_{ij} in the i^{th} row and j^{th} column can be obtained by

$$p_{ij} = p_{i-1,j} + p_{i,j-1} = \binom{i + j - 2}{i - 1} ,$$

and the generating function for the elements appearing in the j^{th} column is $1/(1 - x)^j$, $j = 1, 2, \cdots, n$.

Pascal's triangle written in left-justified form can be imbedded in the $n \times n$ matrix $A = (a_{ij})$,

$$(1.2) \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}_{n \times n}$$

where

$$a_{ij} = \binom{i-1}{j-1},$$

and the column generators are

$$x^{j-1}/(1 - x)^j .$$

The sums of the elements found on the rising diagonals of A, found by beginning at the left-most column and moving right one and up one throughout the array, are 1, 1, 2, 3, 5, 8, 13, $\cdots$, $F_{n+2} = F_{n+1} + F_n$, the Fibonacci numbers.

Theorem 1.1. Any $k \times k$ submatrix of A which contains the column of ones and has for its first row the i^{th} row of A has a determinant of value one.

This is easily proved, for if the preceding row is subtracted from each row successively for $i = k, k-1, \cdots, 2$, and then for $i = k, k-1, \cdots, 3, \cdots$, and finally for $i = k$, the matrix obtained equivalent to the given submatrix has ones on its main diagonal and zeroes below. For example, for $k = 4$ and $i = 4$,

$$\begin{vmatrix} 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \\ 1 & 6 & 15 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 1 & 5 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

which is an interesting process in itself, since each row becomes the same as the first row except moved successively one space right.

Let the $n \times n$ matrix A^T be the transpose of A, so that Pascal's triangle appears on and above the main diagonal with its rows in vertical position. Then $AA^T = P$, the rectangular Pascal's matrix of (1.1). That is, for $n = 4$,

$$AA^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = P .$$

As proof, the generating functions for the columns of A are

$$\frac{1}{1-x} \cdot \left(\frac{x}{1-x}\right)^{j-1} ,$$

and for A^T, $(1 + x)^{j-1}$, $j = 1, 2, \cdots, n$, so that the generating functions for the columns of AA^T are

$$\frac{1}{1-x} \cdot \left(1 + \frac{x}{1-x}\right)^{j-1} = \left(\frac{1}{1-x}\right)^{j}, \qquad j = 1, 2, \cdots, n,$$

which we recognize as the generating functions for the columns of P. Notice that $\det A = \det A^T = 1$, so that $\det AA^T = \det P = 1$ for any n. That is,

Theorem 1.2. The $k \times k$ submatrix, formed from Pascal's triangle written in rectangular form to contain the upper left-hand corner element of P, has a determinant value of one.

In P, if the $(n-1)^{st}$ column is subtracted from the n^{th} column, the $(n-2)^{nd}$ from the $(n-1)^{st}$, $\cdots$, the 2^{nd} from the 3^{rd}, and the first from the 2^{nd}, a new array is formed:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

In the new array, the determinant is still one but it equals the determinant of the $(n-1) \times (n-1)$ array formed by deleting the first row and first column, which array can be found within the original array P by moving one column right and using the original row of ones as the top row. If we again subtract the $(k-1)^{st}$ column from the k^{th} column successively for $k = n-1, n-2, \cdots, 3, 2$, the new array has the determinant value one and can be found as the $(n-2) \times (n-2)$ array using the original row of ones as its top row and two columns right in P. By the law of formation of Pascal's triangle, we can subtract thusly, k times beginning with an $(n+k) \times (n+k)$ matrix P to show that the determinant of any $n \times n$ array within P containing a row of ones has determinant equivalent to $\det P$ and thus has value one. By the symmetry of Pascal's triangle and P, any $n \times n$ square array taken within P to include a column of ones as its edge also has a unit determinant.

We have proved

Theorem 1.3. The determinant of any $n \times n$ array taken with its first row along the row of ones, or with its first column along the column of ones in Pascal's triangle written in rectangular form, is one.

Returning to the matrices A and A^T, AA^T gave us Pascal's triangle in rectangular form. Consider $A^m A^T$

$$A^2A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & \cdots \\ 4 & 8 & 13 & 19 & 26 & \cdots \\ 8 & 20 & 38 & 63 & 96 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the first column is $1, 2, 4, 8, \cdots, 2^k, \cdots$, with column generators

$$\frac{(1 - x)^{j-1}}{(1 - 2x)^j},$$

$j = 1, 2, \cdots, n$. Notice that the sums of elements appearing on the rising diagonals are $1, 3, 8, 21, 55, \cdots$, the Fibonacci numbers with even subscripts.

In general, A^mA^T has first column $1, m, m^2, \cdots, m^k, \cdots$, with column generators

$$\frac{[1 - (m - 1)x]^{j-1}}{(1 - mx)^j}, \quad j = 1, 2, \cdots, n.$$

Notice that any $k \times k$ submatrix of A^mA^T containing the first row (which is a row of ones) has a unit determinant, since that submatrix is the product of submatrices with unit determinants from A and $A^{m-1}A^T$, where $A^1A^T = P$.

In general, if we consider determinants whose rows are composed of successive members of an arithmetic progression of the r^{th} order, we can predict the determinant value as well as give a second proof of the foregoing unit determinant properties of Pascal's triangle. The following theorem is due to Howard Eves [2].

Let us define an <u>arithmetic progression of the r^{th} order</u>, denoted by $(AP)_r$, to be a sequence of numbers whose r^{th} row of differences is a row of constants, but whose $(r - 1)^{th}$ row is not. For example, the third row of Pascal's triangle in rectangular form is an $(AP)_3$, since

$$\begin{array}{ccccccccccccc} 1 & & 4 & & 10 & & 20 & & 35 & & 56 & & \cdots \\ & 3 & & 6 & & 10 & & 15 & & 21 & & & \cdots \\ & & 3 & & 4 & & 5 & & 6 & & & & \cdots \\ & & & 1 & & 1 & & 1 & & & & & \cdots \end{array}$$

A row of repeated constants will be called an $(AP)_0$. The constant in the r^{th} row of differences of an $(AP)_r$ will be called the constant of the progression.

<u>Theorem 1.4.</u> (Eves' Theorem). Consider a determinant of order n whose i^{th} row $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an $(AP)_{i-1}$ with constant a_i. Then the value of the determinant is the product $a_1a_2 \cdots a_n$.

The proof is an armchair one, since, by a sequence of operations of the type where we subtract from a column of the determinant the preceding column, one can reduce the matrix of the determinant to one having $a_1, a_2, \cdots, a_n$ alongthe main diagonal and zeros everywhere above the main diagonal.

Eves' Theorem applies to all the determinants formed from Pascal's triangle discussed in this section, with $a_1 = a_2 = \cdots = a_n = 1$, whence the value of each determinant is one. As a corollary to Eves' Theorem, notice that the n^{th} order determinants containing a row of ones formed from Pascal's rectangular array still have value one if the k^{th} row is shifted m spaces left, $k = 1, 2, \cdots, n$, $m \geq 0$, where m is arbitrary and can be a different value for each row!

Corollary 1.4.1. If the k^{th} row of an $n \times n$ determinant contains n consecutive nonzero members of the $(k-1)^{st}$ row of Pascal's triangle written in rectangular form for each $k = 1, 2, \cdots, n$, the determinant has value one.

Since all arrays discussed in this section contain a row (or column) of ones, we also have infinitely many determinants that can be immediately written with arbitrary value c: merely add $(c-1)$ to each element of the matrix of any one of the unit determinant arrays! The proof is simple: since every element in the first row equals c, factor out c. Then subtract $(c-1)$ times the first row from each other row, returning to the original array which had a determinant value of one. Thus, the new determinant has value c. This is a special case of a wide variety of problems concerning the lambda number of a matrix [3], [4].

Eves' Theorem also applies to the arrays of convolution triangles and multinomial coefficients which follow, but the development using other methods of proof is more informative.

2. OTHER DETERMINANT VALUES FROM PASCAL'S TRIANGLE

Return again to matrix P of (1.1). Suppose that we remove the top row and left column, and then evaluate the $k \times k$ determinants containing the upper left corner. Then

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \qquad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \qquad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4,$$

and the $k \times k$ determinant has value $(k+1)$.

Proof is by mathematical induction. Assume that the $(k-1) \times (k-1)$ determinant has value k. In the $k \times k$ determinant, subtract the preceding column from each column successively for $j = k, k-1, k-2, \cdots, 2$. Then subtract the preceding row from each row successively for $i = k, k-1, k-2, \cdots, 2$, leaving

$$\begin{vmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & \cdots \\ 0 & 3 & 6 & 10 & \cdots \\ 0 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 1 + k.$$

Returning to matrix P, take 2×2 determinants along the second and third rows:

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \cdots,$$

giving the values found in the second column of Pascal's left-justified triangle, for

$$\begin{vmatrix} \binom{j}{1} & \binom{j+1}{1} \\ \binom{j+1}{2} & \binom{j+2}{2} \end{vmatrix} = \binom{j+1}{2}$$

by simple algebra. Of course, 1×1 determinants along the second row of P yield the successive values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10,$$

the successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the 2^{nd}, 3^{rd}, $\cdots$, and $(k+1)^{st}$ rows yields the successive entries of the k^{th} column of Pascal's triangle.

To formalize our statement,

Theorem 2.1: The determinant of the $k \times k$ matrix $R(k,j)$ taken with its first column the j^{th} column of P, the rectangular form of Pascal's triangle imbedded in a matrix, and its first row the second row of P, is the binomial coefficient

$$\binom{j-1+k}{k}.$$

To illustrate,

$$\det R(4,3) = \begin{vmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 6 & 4 & 5 & 6 \\ 10 & 10 & 15 & 21 \\ 15 & 20 & 35 & 56 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 \\ 0 & 6 & 10 & 15 \\ 0 & 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \\ 5 & 15 & 35 & 70 \end{vmatrix}$$

$$= \det R(3,3) + \det R(4,2)$$

$$= \binom{5}{3} + \binom{5}{4} = \binom{6}{4} = 15\,.$$

First, the preceding column was subtracted from each column successively, $j = k, k-1, \cdots, 2$, and then the preceding row was subtracted from each row successively for $i = k, k-1, \cdots, 2$. Then the determinant was made the sum of two determinants, one bordering $R(3,3)$ and the other equal to $R(4,2)$ by adding the j^{th} column to the $(j+1)^{st}$, $j = 1, 2, \cdots, k-1$.

By following the above procedure, we can make

$$\det R(k,j) = \det R(k-1, j) + \det R(k, j-1)\,.$$

We have already proved that

$$\det R(k,1) = 1 = \binom{k+0}{k}, \qquad \det R(k,2) = k+1 = \binom{k+1}{k} \quad \text{for all } k,$$

$$\det R(1,j) = j = \binom{j+0}{1}, \qquad \det R(2,j) = \binom{j+1}{2} \quad \text{for all } j.$$

If

$$\det R(k-1, j) = \binom{j+k-2}{k-1} \quad \text{and} \quad \det R(k, j-1) = \binom{j+k-2}{k},$$

then

$$\det R(k,j) = \binom{j+k-2}{k-1} + \binom{j+k-2}{k} = \binom{j+k-1}{k}$$

for all k and all j by mathematical induction.

Since P is its own transpose, Theorem 2.1 is also true if the words "column" and "row" are everywhere exchanged.

Consider Pascal's triangle in the configuration of A^T, which is just Pascal's rectangular array P with the i^{th} row moved $(i - 1)$ spaces right, $i = 1, 2, 3, \cdots$. Form $k \times k$ matrices $R'(k, j)$ such that the first row of $R'(k, j)$ is the second row of A^T beginning with the j^{th} column of A^T. Then $AR'(k, j - 1) = R(k, j)$ as can be shown by considering their column generating functions, and since $\det A = 1$, $\det R'(k, j - 1) = \det R(k, j)$, leading us to the following theorems.

Theorem 2.2. Let A^T be the $n \times n$ matrix containing Pascal's triangle on and above its main diagonal so that the rows of Pascal's triangle are placed vertically. Any $k \times k$ submatrix of A^T selected with its first row along the second row of A^T and its first column the j^{th} column of A^T, has determinant value

$$\binom{k+j-2}{k} .$$

Since A is the transpose of A^T, wording Theorem 2.2 in terms of the usual Pascal triangle provides the following.

Theorem 2.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the i^{th} row has determinant value given by the binomial coefficient

$$\binom{k+i-1}{k} .$$

3. MULTINOMIAL COEFFICIENT ARRAYS

The trinomial coefficients arising in the expansions of $(1 + x + x^2)^n$, written in left-justified form, are

(3.1)

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
...	...	...	...	...	...	...	...	...	...	...	...	...

where the entry t_{ij} in the i^{th} row and j^{th} column is obtained by the relationship

$$t_{ij} = t_{i-1,j} + t_{i-1,j-1} + t_{i-1,j-2} .$$

The sums of elements on rising diagonals formed by beginning in the left-most column and counting up one and right one are the Tribonacci numbers 1, 1, 2, 4, 7, 13, ⋯, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. (As in Pascal's triangle, the left-most column is the zeroth column and the top row the zeroth row.)

If the summands for the Fibonacci numbers from the rising diagonals of Pascal's triangle (1.2) in left-justified form are used in reverse order as the rows to form an $n \times n$ matrix

$$F_1 = (f_{ij}), \qquad f_{ij} = \binom{j-1}{i-j},$$

which has the rows of Pascal's triangle written in vertical position on and below the main diagonal, and the matrix A^T is written with Pascal's triangle on and above the main diagonal, then the matrix product F_1A^T is the trinomial coefficient array (3.1) but written vertically rather than in the horizontal arrangement just given. To illustrate, when $n = 7$,

$$F_1A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 \\ 0 & 0 & 0 & 0 & 1 & 5 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 \\ 0 & 0 & 2 & 7 & 16 & 30 & 50 \\ 0 & 0 & 1 & 6 & 19 & 45 & 90 \\ 0 & 0 & 0 & 3 & 16 & 51 & 126 \\ 0 & 0 & 0 & 1 & 10 & 45 & 141 \end{bmatrix} .$$

Generating functions give an easy proof. The generating function for the j^{th} column of F_1 is $[x(1+x)]^{j-1}$, $j = 1, 2, \cdots$. The generating function for the j^{th} column of A^T is $(1+x)^{j-1}$, $j = 1, 2, \cdots$, so the generating function for the j^{th} column of F_1P is $[1 + x(1+x)]^{j-1}$ or $(1 + x + x^2)^{j-1}$, so that F_1A^T has the rows of the trinomial triangle as its columns. Since $\det F_1 = \det A^T = 1$, $\det F_1A^T = 1$, and $n \times n$ determinants containing the upper left corner of the trinomial triangle have value one. Further, any $k \times k$ submatrix of F_1A^T containing a row of ones has a unit determinant, for any $k \times k$ submatrix of F_1A^T selected with a row of ones is always the product of a submatrix of F_1 and a submatrix of A^T containing a row of ones, both of which here have unit determinants. Lastly, any $k \times k$ submatrix of F_1A^T having its first row along the second row of F_1A^T and its first column the j^{th} column of F_1A^T is the product of a submatrix of F_1 with a unit determinant and a $k \times k$ submatrix of A^T satisfying Theorem 2.2, and thus has determinant value

$$\binom{k+j-2}{k} .$$

We can form matrices F_m analogous to F_1 using the rising diagonals of the triangle of multinomial coefficients arising from the expansions of $(1 + x + x^2 + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, as the rows of F_m, so that the rows of the multinomial triangle appear as the columns of F_m written on and below the main diagonal. (In each case, the row sums of F_m are $1 = r_1 = r_2 = \cdots = r_m$, $r_n = r_{n-1} + r_{n-2} + \cdots + r_{n-m}$, $n \geq m + 1$.) The matrix product F_mA^T will contain the coefficients arising from the expansions of

$$(1 + x + x^2 + \cdots + x^{m+1})^n$$

written in vertical position on and above the main diagonal. As proof, the generating functions of columns of F_m are $[x(1 + x + \cdots + x^m)]^{j-1}$, $j = 1, 2, \cdots, n$, while the generating function for the j^{th} column of A^T is $(1 + x)^{j-1}$, $j = 1, 2, \cdots, n$, so that the generating function for the j^{th} column of F_mA^T is

$$[1 + (x(1 + x + \cdots + x^m))]^{j-1} = [1 + x + x^2 + \cdots + x^{m+1}]^{j-1}, \quad j = 1, 2, \cdots, n .$$

If a submatrix of F_mA^T is the product of a submatrix of F_m with unit determinant and a submatrix of $F_{m-1}A^T$ taken in the corresponding position, it will thus have the same determinant as the submatrix of $F_{m-1}A^T$. Notice that this is true for submatrices taken across the first or second columns of F_mA^T. Since F_1A^T has its submatrices with the same determinant values as the submatrices of A^T taken in corresponding position, and these values are given in Theorem 2.2, we have proved the following theorems by mathematical induction if we look at the transpose of F_mA^T.

Theorem 3.1. If the multinomial coefficients arising from the expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, are written in left-justified form the determinant of the $k \times k$ matrix formed with its first column taken anywhere along the leftmost column of ones of the array is one.

Theorem 3.2. If the multinomial coefficients arising from the expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, are written in left-justified form the determinant of the $k \times k$ matrix formed with its first column the first column of the array (the column of successive whole numbers) and its first row the i^{th} row of the multinomial coefficient array, has determinant value given by the binomial coefficient

$$\binom{k+i-1}{k} .$$

4. THE FIBONACCI CONVOLUTION ARRAY AND RELATED CONVOLUTION ARRAYS

If $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ are two sequences, the convolution sequence $\{c_i\}_{i=1}^{\infty}$ is given by

$$c_1 = a_1b_1, \qquad c_2 = a_2b_1 + a_1b_2, \qquad c_3 = a_3b_1 + a_2b_2 + a_1b_3, \cdots,$$

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} .$$

If $g(x)$ is the generating function for $\{a_i\}$, then $[g(x)]^{k+1}$ is the generating function for the k^{th} convolution of $\{a_i\}$ with itself.

The Fibonacci sequence, when convoluted with itself $j - 1$ times, forms the sequence in the j^{th} column of the matrix $C = (c_{ij})$ below [1], where the original sequence is in the leftmost column

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 3 & 10 & 22 & 40 & 65 & 98 & \cdots \\ 5 & 20 & 51 & 105 & 190 & 315 & \cdots \\ 8 & 38 & 111 & 256 & 511 & 924 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n} .$$

For formation of the Fibonacci convolution array,

$$c_{ij} = c_{i-1,j} + c_{i-2,j} + c_{i,j-1} .$$

Let the $n \times n$ matrix F_1 be formed as in Section 3 with the rows of Pascal's triangle in vertical position on and below the main diagonal, and let the $n \times n$ matrix P be Pascal's rectangular array (1.1). That F_1P is the Fibonacci convolution array in rectangular form is easily proved. The generating functions for the columns of P_1 are $[x(1 + x)]^{j-1}$, while those for P are $1/(1 - x)^j$, $j = 1, 2, \cdots, n$, so that the generating functions for the columns of F_1P are $1/[1 - x(1 + x)]^j$ or $1/(1 - x - x^2)^j$, the generators of C, the Fibonacci convolution array. Since $\det F_1 = \det P = 1$, $\det F_1P = 1$, and any $n \times n$ matrix formed using the upper left corner of the Fibonacci convolution array has a unit determinant.

Further, since submatrices of C taken along either the first or second row of C are the product of submatrices of F_1 with unit determinants and similarly placed submatrices of P whose determinants are given in Theorems 1.3 and 2.1, we have the following theorem.

<u>Theorem 4.1.</u> Let the Fibonacci convolution triangle be written in rectangular form and imbedded in an $n \times n$ matrix C. Then the determinant of any $k \times k$ submatrix of C placed

to contain the row of ones is one. Also, the determinant of any $k \times k$ submatrix of C selected with its first row along the second row of C and its first column the j^{th} column of C has determinant value given by the binomial coefficient $\binom{k+j-1}{k}$.

The generalization to convolution triangles for sequences which are found as sums of rising diagonals of multinomial coefficient triangles written in left-justified form is not difficult.

Form the matrix F_m as in Section 4 to have its rows the elements found on the rising diagonals of the left-justified multinomial coefficient triangle induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$. Then F_mP is the convolution triangle for the sequence of sums of the rising diagonals just described, for F_m has column generators $[x(1 + x + \cdots + x^m)]^{j-1}$ and P has column generators $1/(1-x)^j$, making the column generators of F_mP be $1/(1 - x - x^2 - \cdots - x^{m+1})^j$, $j = 1, 2, \cdots, n$, which for $j = 1$ is known to be the generating function for the sequence of sums found along the rising diagonals of the given multinomial coefficient triangle [5].

As before, since a submatrix of F_mP is the product of a submatrix of F_m with unit determinant and a similarly placed submatrix of P with determinant given by Theorems 1.3 and 2.1 when we take submatrices along the first or second column of F_mP, we can write the following.

Theorem 4.2. Let the convolution triangle for the sequences of sums found along the rising diagonals of the left-justified multinomial coefficient array induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, be written in rectangular form and imbedded in an $n \times n$ matrix C^*. Then the determinant of any $k \times k$ submatrix of C^* placed to contain the row of ones is one. Also, the determinant of any $k \times k$ submatrix of C^* selected with its first row along the second row of C^* and its first column the j^{th} column of C^* has determinant value given by the binomial coefficient $\binom{k+j-1}{k}$.

The convolution triangle imbedded in the matrix $C^* = (c_{ij})$ of Theorem 4.2 has $c_{ij} = c_{i-1,j} + c_{i-2,j} + \cdots + c_{i-m,j} + c_{i,j-1}$. If we form an array $B = (b_{ij})$ with rule of formation $b_{ij} = b_{i-1,j} + b_{i-2,j} + \cdots + b_{2,j} + b_{1,j} + b_{i,j-1}$, B becomes the powers of 2 convolution array.

1	1	1	1	1	1	⋯
1	2	3	4	5	6	⋯
2	5	9	14	20	27	⋯
4	12	25	44	70	104	⋯
8	28	66	129	225	463	⋯
16	64	168	360	681	1291	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯

Notice in passing that the rising diagonal sums are 1, 2, 5, 13, 34, ⋯, the Fibonacci numbers with odd subscripts. This convolution array has both unit determinant properties discussed in Section 1 because it fulfills Eves' Theorem. In fact, all the convolution arrays of this section fulfill Eves' Theorem, so that any $k \times k$ matrix placed to contain a row of ones in any one of these convolution arrays has determinant value one.

5. CONVOLUTION ARRAYS FOR DIAGONAL SEQUENCES FROM MULTINOMIAL COEFFICIENT ARRAYS

Let the $n \times n$ matrix D be

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 4 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the row sums of D are the rising diagonal sums from Pascal's triangle (1.2) found by beginning at the leftmost column and going up 2 and to the right one, namely, 1, 1, 1, 2, 3, 4, 6, 9, 13, $\cdots$. The column generators of D are $[x(1+x^2)]^j$, $j = 1, 2, \cdots$. Then, the matrix product DP, where P is the rectangular Pascal matrix (1.1), is the convolution triangle for the sequence 1, 1, 1, 2, 3, 4, 6, 9, 13, $\cdots$, $u_{n+1} = u_n + u_{n-2}$, for the column generators of DP are $1/[1 - x(1+x^2)]^j = 1/(1 - x - x^3)^j$, $j = 1, 2, \cdots, n$, where the generating function for $j = 1$ is given as the generating function for the sequence discussed in [5]. The results of Theorem 1.3 and 2.1 can be extended to cover the matrix DP as before.

Now, consider the sequence of elements in Pascal's triangle that lie on the diagonals found by beginning at the leftmost column of (1.2) and going up p and to the right 1 throughout the array (1.2). (The sum of the elements on these diagonals are the generalized Fibonacci numbers $u(n; p, 1)$ of Harris and Styles [6].) Place the elements from successive diagonals on rows of a matrix $D_1(p,1)$ in reverse order such that the column of ones in A lies on the diagonal of $D_1(p,1)$. Then the columns of $D_1(p,1)$ are the rows of Pascal's triangle in left-justified form but with the entries separated by $(p - 1)$ zeroes. Notice that $\det D_1(p,1) = 1$. The column generators of $D_1(p,1)$ are $[x(1+x^p)]^j$, $j = 1, 2, \cdots$, so that the column generators of $D_1(p,1)P$ are $1/[1 - x(1+x^p)]^j = 1/(1 - x - x^{p+1})^j$, $j = 1, 2, \cdots, n$, the generating functions for the convolution array for the numbers $u(n; p, 1)$ (see [5]). The same arguments hold for $D_1(p,1)P$ as for $DP = D_1(2,1)P$, so that $D_1(p,1)P$ has the determinant properties which extend from Theorems 1.3 and 2.1.

Now, the same techniques can be applied to the elements on the rising diagonals in all the generalized Pascal triangles induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \ge 1$, $n \ge 0$. Form the $n \times n$ matrix $D_m(p,1)$ so that elements on the diagonals formed by beginning in the leftmost column and going up p and right one throughout the left-justified multinomial coefficient array lie in reverse order on its rows. $D_m(p,1)$ will have a one for each element on its main diagonal and each column will contain the corresponding row of the multinomial array but with $(p - 1)$ zeroes between entries. Generating functions for the columns of

$D_m(p,1)$ are $[x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^j$, $j = 1, 2, \cdots, n$, while for P they are, of course, $1(1-x)^j$. The generating functions for the columns of $D_m(p,1)P$ are then

$$1/[1 - x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^2$$

or

$$1/[1 - x - x^{p+1} - x^{2p+1} - \cdots - x^{(m-1)p+1}]^j, \quad j = 1, 2, \cdots, n,$$

which, for $j = 1$, is known to be the generating function for the diagonal sums here considered [5].

Since again the $k \times k$ submatrices of the product $D_m(p,1)P$ taken along either the first or second row of $D_m(p,1)P$ have the same determinants as the correspondingly placed $k \times k$ submatrices of P, we look again at Theorems 1.3 and 2.1 to write our final theorem.

Theorem 5.1. Write the convolution triangle in rectangular form imbedded in an $n \times n$ matrix C_m^* for the sequence of sums found on the rising diagonals formed by beginning in the leftmost column and moving up p and right one throughout any left-justified multinomial coefficient array. The $k \times k$ submatrix formed to include the first row of ones has determinant one. The $k \times k$ submatrix formed with its first row the second row of C_m^* and its first column the j^{th} column of C_m^* has determinant given by the binomial coefficient

$$\binom{k+j-1}{k} .$$

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