THROUGH THE OTHER END OF THE TELESCOPE

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Base two has this interesting property that all integers may be represented uniquely by a sequence of zeros and ones. If instead of starting with base two, we had started with the sequence of ones and zeros and correlated the integers with them, then we would have seen that it is powers of two that correspond to a representation one followed by a number of zeros. This is what is meant in the title by looking through the other end of the telescope.

Table 1
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS

Representations	Integers	Representations	Integers
1	1	1001	9
10	2	1010	10
11	3	1011	11
100	4	1100	12
101	5	1101	13
110	6	1110	14
111	7	1111	15
1000	8	10000	16

If we continue this sequence of ones and zeros, will a one followed by zeros always be a power of two? Yes it will. For example, the four zeros in the representation of 16 will take on all the changes from 0001 to 1111 and bring us to 31 so that 100000 will be 32. In general, if there is a one followed by r zeros representing 2^r the last number that can be represented before increasing the number of digits will be:

$$2^r + (2^r - 1) = 2^{r+1} - 1 .$$

Thus, the next representation which is a 1 followed by $r + 1$ zeros will represent 2^{r+1}.

But is there anything particularly sacred about the way our sequence of ones and zeros has been chosen? Must it even be that the ones in various positions must represent the power of a number?

Suppose we change the rules for creating our succession of representations by insisting that no two ones be adjacent to each other.

Table 2
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS, NO TWO ONES ADJACENT TO EACH OTHER

Representations	Integers	Representations	Integers	Representations	Integers
1	1	101010	20	10001000	39
10	2	1000000	21	10001001	40
100	3	1000001	22	10001010	41
101	4	1000010	23	10010000	42
1000	5	1000100	24	10010001	43
1001	6	1000101	25	10010010	44
1010	7	1001000	26	10010100	45
10000	8	1001001	27	10010101	46
10001	9	1001010	28	10100000	47
10010	10	1010000	29	10100001	48
10100	11	1010001	30	10100010	49
10101	12	1010010	31	10100100	50
100000	13	1010100	32	10100101	51
100001	14	1010101	33	10101000	52
100010	15	10000000	34	10101001	53
100100	16	10000001	35	10101010	54
100101	17	10000010	36	100000000	55
101000	18	10000100	37		
101001	19	10000101	38		

It is a matter of observation from this table that one followed by zeros is a Fibonacci number. If we take the series as $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $\cdots$ then the one in the r^{th} place from the right represents F_{r+1}.

Will this continue? Consider one followed by nine zeros or F_{10}. Since there may not be a one next to the first one, the numbers added to F_{10} in the succeeding representations are all the numbers up to and including 33, so that the final sum can be represented with ten digits is $55 + 34 - 1 = 89 - 1$. Thus one followed by ten zeros is 89 or F_{11}. A similar argument can be applied in general.

What happens if we insist that no two ones have less than two zeros between them? Again we can form a table. (See Table 3.) The sequence of integers that correspond to one followed by zeros is: 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, $\cdots$. Is there a law of formation of the sequence? It appears that

$$
\begin{aligned}
9 &= 6 + 3 \\
13 &= 9 + 4 \\
19 &= 13 + 6 \\
28 &= 19 + 9 \\
41 &= 28 + 13
\end{aligned}
$$

Table 3
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS, NO TWO ONES SEPARATED BY LESS THAN TWO ZEROS

Representations	Integers	Representations	Integers	Representations	Integers
1	1	1000010	15	100000001	29
10	2	1000100	16	100000010	30
100	3	1001000	17	100000100	31
1000	4	1001001	18	100001000	32
1001	5	10000000	19	100001001	33
10000	6	10000001	20	100010000	34
10001	7	10000010	21	100010001	35
10010	8	10000100	22	100010010	36
100000	9	10001000	23	100100000	37
100001	10	10001001	24	100100001	38
100010	11	10010000	25	100100010	39
100100	12	10010001	26	100100100	40
1000000	13	10010010	27	1000000000	41
1000001	14	100000000	28		

or if the terms of the sequence are denoted by T_n,

$$T_{n+1} = T_n + T_{n-2} .$$

Will this continue? If we go beyond 41 the largest number that can be represented before increasing the number of digits is 1000000000 plus 1001001. Since this puts a 1 three places beyond the first 1 and is the largest number that can be represented of this type. Hence one followed by 10 zeros is 41 + 19 or 60. Evidently the argument can be applied in general.

Going one step further, we set the condition that two ones may not have less than three zeros between them.

Table 4
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS, NO TWO ONES SEPARATED BY LESS THAN THREE ZEROS

Representations	Integers	Representations	Integers	Representations	Integers
1	1	100001	8	10000001	15
10	2	100010	9	10000010	16
100	3	1000000	10	10000100	17
1000	4	1000001	11	10001000	18
10000	5	1000010	12	100000000	19
10001	6	1000100	13	100000001	20
100000	7	10000000	14	100000010	21

(Table continues on the following page.)

Table 4 (Continued)

Representations	Integers	Representations	Integers	Representations	Integers
100000100	22	1000000001	27	1000010001	32
100001000	23	1000000010	28	1000100000	33
100010000	24	1000000100	29	1000100001	34
100010001	25	1000001000	30	1000100010	35
1000000000	26	1000010000	31	10000000000	36

We note that

$$14 = 10 + 4$$
$$19 = 14 + 5$$
$$26 = 19 + 7$$
$$36 = 26 + 10$$

suggesting the relation

$$T_{n+1} = T_n + T_{n-3} .$$

The following table summarizes the situation out to the case in which two ones may not have less than six zeros between them (system denoted S_6).

Table 5
NUMBERS REPRESENTED BY A UNIT IN THE n^{th} PLACE FROM THE LEFT FOR VARIOUS ZERO SPACINGS

n	S_0	S_1	S_2	S_3	S_4	S_5	S_6
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	4	3	3	3	3	3	3
4	8	5	4	4	4	4	4
5	16	8	6	5	5	5	5
6	32	13	9	7	6	6	6
7	64	21	13	10	8	7	7
8	128	34	19	14	11	9	8
9	256	55	28	19	15	12	10
10	512	89	41	26	20	16	13
11	1024	144	60	36	26	21	17
12	2048	233	88	50	34	27	22
13	4096	377	129	69	45	34	28
14	8192	610	189	95	60	43	35
15	16384	987	277	131	80	55	43
16	32768	1597	406	181	106	71	53

(Table continues on following page.)

Table 5 (Continued)

n	S_0	S_1	S_2	S_3	S_4	S_5	S_6
17	65536	2584	595	250	140	92	66
18	131072	4181	872	345	185	119	83
19	262144	6765	1278	476	245	153	105
20	524288	10946	1873	657	325	196	133

To represent a given number in any one of these systems it is simply necessary to keep subtracting out the largest number less than or equal to the remainder. Thus to represent 342 (base 10) in S_4, we proceed as follows:

$$342 - 325 = 17$$
$$17 - 15 = 2 .$$

The representation is 10000000000100000010. Representations of 342 in all the systems are as follows.

S_0	101010110
S_1	101000101010
S_2	100010000001001
S_3	10001000100001000
S_4	10000000000100000010
S_5	1000000000001000001000
S_6	1000000000000000100000010

GENERATING FUNCTIONS OF THESE SYSTEMS

The following are somewhat more advanced considerations for the benefit of those who can pursue them. A generating function as employed here is an algebraic expression which on being developed into an infinite power series has for coefficients the terms of a given sequence. Thus for S_0, it can be found by a straight process of division that formally:

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots .$$

For S_1, the Fibonacci sequence, it is known that:

$$\frac{1+x}{1-x-x^2} = F_2 + F_3x + F_4x^2 + F_5x^3 + \cdots .$$

The process of determining such coefficients may be illustrated by this case. Set

$$\frac{1+x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots ,$$

so that on multiplying through by $1 - x - x^2$,

$$1 + x = (1 - x - x^2)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) \quad .$$

This must be an identity so that the coefficients of the powers of x on the left-hand side must equal the coefficients of the corresponding powers of x on the right-hand side. Thus:

$$\begin{aligned} a_0 &= 1 & & \\ a_1 - a_0 &= 1, & \text{so that} \quad a_1 &= 2 \\ a_2 - a_1 - a_0 &= 0, & \text{so that} \quad a_2 &= 3 \\ a_3 - a_2 - a_1 &= 0, & \text{so that} \quad a_3 &= 5 \end{aligned}$$

and since in general $a_n - a_{n-1} - a_{n-2} = 0$, it is clear that the Fibonacci relation holds for successive sets of terms of the sequence, so that the Fibonacci numbers must continue to appear in order with $a_n = F_{n+2}$.

On the basis of the initial terms of the sequence and the type of recursion relation involved, the generating function for S_2 should be:

$$\frac{1 + x + x^2}{1 - x - x^3} ,$$

which can be verified in the same way as for S_1.

In general for S_k, the generating function would be:

$$\frac{1 + x + x^2 + \cdots + x^k}{1 - x - x^{k+1}} .$$

CONCLUSION

There is an endless sequence of number representations involving only ones and zeros with the following properties:

1. In each system, every number has a unique representation.
2. In the system S_k (two ones separated by not less than k zeros), the recursion relation connecting the numbers represented by units in the various positions is:

$$T_{n+1} = T_n + T_{n-k} .$$

3. The well known unique representations in base 2 and by means of non-adjacent Fibonacci numbers (Zeckendorf's Theorem) are the first two of these number representations, namely, S_0 and S_1.