GENERALIZED FIBONACCI SHIFT FORMULAS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

A Fibonacci sequence is one which is governed by the relation

$$T_{n+1} = T_n + T_{n-1} \tag{1}$$

among successive terms. From this simple beginning we arrive at numerous recursion relations for related sequences of various types. These form the starting points for the generalized Fibonacci shift formulas to be treated in this paper.

Given any number of Fibonacci sequences such as:

(2)
F_n: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ⋯
L_n: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, ⋯
T_n: 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, 1741, 2817, 4558, ⋯

We can consider terms such as F_{n-3}, L_{n+4}, T_{n-1} and combine these to form a Fibonacci expression of some degree. For example

$$F^2_{n-3} L_{n+4} T_{n-1} + L^4_{n+4} - T^3_{n-1} F_{n-3}$$

would be spoken of as a homogeneous Fibonacci expression of the fourth degree. But we might also have subsets of terms of these sequences in which only every other term is involved in successive values of the given expression. Such terms would be of the form F_{2n+a}, L_{2n+b}, T_{2n+c}, where a, b, and c are fixed constants. In general, if terms are of the form F_{mn+a}, L_{mn+b}, T_{mn+c}, speak of such terms as being of class m.

Now the recursion relations for homogeneous expressions of any degree for a given class are Fibonacci coefficients which are built up from certain subsequences of the Fibonacci sequence (F_n). For class 1 (m = 1), the Fibonomial coefficients are formed from F_1, F_2, F_3, F_4, ⋯. In analogy to the binomial coefficients

$$\binom{r}{j} = \frac{r!}{j!\,(r-j)!} = r_j/j! \quad , \tag{3}$$

where $r_j = r(r-1)(r-2) \cdots (r-j+1)$, we have for the Fibonomial coefficients

(4) $$F[r,j] = \frac{[F_r]!}{[F_j]!\,[F_{r-j}]!}$$

$$= \frac{F_r F_{r-1} F_{r-2} \cdots F_2 F_1}{F_j F_{j-1} F_{j-2} \cdots F_2 F_1 F_{r-j} F_{r-j-1} \cdots F_2 F_1}$$

$$= [F_r]_j / [F_j]! \quad ,$$

where

$$[F_r]_j = F_r F_{r-1} \cdots F_{r-j+1} \ .$$

For example,

$$F[10,4] = \frac{F_{10} F_9 F_8 F_7}{F_1 F_2 F_3 F_4} = 85085 \quad .$$

For class 2 $(m = 2)$, the Fibonomial coefficients are formed from F_2, F_4, F_6, F_8, F_{10}, $\cdots$. And in general for class m, the Fibonomial coefficients are formed from F_m, F_{2m}, F_{3m}, F_{4m}, $\cdots$. (For background on the Fibonomial coefficients and their relation to Fibonacci recursion formulas see references 1, 2, 3, 4, 5, and 6.)

To distinguish the various types of Fibonomial coefficients, we shall introduce the following symbolism.

$$F[\text{class, order, index}] = F[m,r,n] \quad .$$

Thus the previously given $F[10,4]$ would be written $F[1,10,4]$. As another example

$$F[2,8,3] = (F_{16} F_{14} F_{12})/(F_2 F_4 F_6) = 2232594 \ .$$

It may be noted that $F[m,r,0] = 1$ by definition.

We need as well symbolism for homogeneous Fibonacci expressions of class m, degree d and running subscript n. These will be denoted

$$H[\text{class, degree, subscript}] = H[m,d,n] \ .$$

Thus we might have

$$H[3,4,n] = F_{3n} L_{3n+1}^2 T_{3n-2} - F_{3n+2}^4 \ .$$

The starting point of our shift formulas can be given as follows. For m even,

(5) $$H[m,\, d,\, n+1] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m,\, d+1,\, i] H[m,\, d,\, n+1-i] \ .$$

For m odd,

$$(6) \qquad H[m, d, n+1] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m, d+1, i] H[m, d, n+1-i] \quad ,$$

where the square brackets in the exponent signify the greatest integer function. Upward shift formulas can be derived from (5) and (6). For m even, we have

$$(7) \qquad H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m, d+1, i] H[m, d, n+i-1]$$

For m odd and d odd,

$$(8a) \qquad H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{[(i+2)/2]} F[m, d+1, i] H[m, d, n+i-1]$$

For m odd and d even,

$$(8b) \qquad H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m, d+1, i] H[m, d, n+i-1] \quad .$$

It may be noted that in all the formulas (5) through (8) the coefficients are numerically the same differing only in sign.

The shift formulas considered in this article give $H[m, d, n+k]$ in terms of $H[m, d, n]$, $H[m, d, n-1]$, $\cdots$ for a downward shift of k. A second set of formulas give $H[m, d, n-k]$ in terms of $H[m, d, n]$, $H[m, d, n+1]$, $\cdots$ for an upward shift of k.

We shall proceed by examining the linear case for various classes, then expressions of the second degree, and so on until formulas applying to all degrees and classes emerge.

FIRST DEGREE. $m = 1$

$$H[1, 1, n+1] = H[1, 1, n] + H[1, 1, n-1] \quad .$$

We substitute $H[1, 1, n] = H[1, 1, n-1] + H[1, 1, n-2]$. To avoid a great deal of writing the following scheme showing only coefficients will be employed.

k = 1	1	1				
		1	1			Substitution line
k = 2		2	1			
			2	2		
k = 3			3	2		
				3	3	
k = 4				5	3	

It appears that $H[1, 1, n+1] = F_{k+1}H[1, 1, n] + F_kH[1, 1, n-1]$ a well known Fibonacci shift formula.

In the other direction $H[1, 1, n-1] = -H[1, 1, n] + H[1, 1, n+1]$. Schematically

$$\begin{array}{lrrrrr} k = 1 & -1 & 1 & & & \\ & & \underline{+1} & \underline{-1} & & \\ k = 2 & & 2 & -1 & & \\ & & & \underline{-2} & \underline{2} & \\ k = 3 & & & -3 & 2 & \\ & & & & \underline{3} & \underline{-3} \\ k = 4 & & & & 5 & -3 \end{array}$$

leading to the relation

$$H[1, 1, n-k] = (-1)^k \{F_{k+1} H[1, 1, n] - F_k H[1, 1, n+1]\}$$

<u>m = 2</u>

The initial relation is

$$H[2, 1, n+1] = 3H[2, 1, n] - H[2, 1, n-1] \quad .$$

In terms of coefficients

$$\begin{array}{lrrrrr} k = 1 & 3 & -1 & & & \\ & & \underline{9} & \underline{-3} & & \\ k = 2 & & 8 & -3 & & \\ & & & \underline{24} & \underline{-8} & \\ k = 3 & & & 21 & -8 & \\ & & & & \underline{63} & \underline{-21} \\ k = 4 & & & & 55 & -21 \end{array}$$

Generalizing

$$H[2, 1, n+k] = F_{2k+2} H[2, 1, n] - F_{2k} H[2, 1, n-1] \quad .$$

For $m = 3$, coefficients are as follows:

k = 1	4	1
k = 2	17	4
k = 3	72	17
k = 4	305	72

To identify these quantities, consult a table of Fibonomial coefficients (see [7], for example) for $m = 2$. Then it can be seen that

$$H[3, 1, n+k] = F[3, k+1, 1]H[3, 1, n] + F[3, k, 1]H[3, 1, n-1] \quad .$$

To conclude, the formulas that apply in the first degree case are as follows.

<u>m odd</u>

$$H[m, 1, n+k] = F[m, k+1, 1]H[m, 1, n] + F[m, k, 1]H[m, 1, n-1]$$

$$H[m, 1, n-k] = (-1)^k \{F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n+1]\}$$

<u>m even</u>

$$H[m, 1, n+k] = F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n-1]$$

$$H[m, 1, n-k] = F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n+1]$$

SECOND DEGREE

As an example of the way in which the coefficients are calculated consider the case $m = 3$ and downward shift.

$k = 1$	17	17	-1			
		289	289	-17		
$k = 2$		306	288	-17		
			5202	5202	-306	
$k = 3$			5490	5185	-306	
				93330	93330	-5490
$k = 4$				98515	93024	-5490

The leading quantity identifies as $F[3, k+2, 2]$, the final quantity as $F[3, k+1, 2]$. The middle quantity is $F[3, k+2, 1]\,F[3, k, 1]$.

The general formulas for the second degree are as follows:

m odd

$$H[m, 2, n+k] = F[m, k+2, 2]H[m,2,n] + F[m, k+2, 1]F[m,k,1]H[m, 2, n-1] - F[m, k+1, 2]H[m, 2, n-2]$$

$$H[m, 2, n-k] = F[m, k+2, 2]H[m,2,n] + F[m, k+2, 1]F[m,k,1]H[m, 2, n+1] - F[m, k+1, 2]H[m, 2, n+2] .$$

m even

$$H[m, 2, n+k] = F[m, k+2, 2]H[m, 2, n] - F[m, k+2, 1]F[m,k,1]H[m, 2, n-1] + F[m, k+1, 2]H[m, 2, n-2]$$

$$H[m, 2, n-k] = F[m, k+2, 2]H[m,2,n] - F[m, k+2, 1]F[m,k,1]H[m, 2, n+1] + F[m, k+1, 2]H[m, 2, n+2] .$$

For the third and fourth degrees, the results are given only for the downward shift case since the coefficients in the upward shift case are numerically the same.

THIRD DEGREE

m odd

$$H[m, 3, n+k] = F[m, k+3,3]H[m, 3, n] + F[m, k+3, 2]F[m, k, 1]H[m, 3, n-1] - F[m, k+3,1]F[m, k+1,2]H[m, 3, n-2] - F[m, k+2,3]H[m, 3, n-3] .$$

m even

Same coefficients and functions with signs $+ - + -$.

FOURTH DEGREE

m odd

$$H[m,4, n+k] = F[m, k+4,4]H[m,4,n] + F[m, k+4,3]F[m,k,1]H[m, 4, n-1] - F[m, k+4,2]F[m, k+1,2]H[m,4, n-2] - F[m, k+4,1]F[m, k+2,3]H[m,4, n-3] + F[m, k+3,4]H[m, 4, n-4].$$

m even

Same expression with alternating signs.

The situation can be summarized as follows as a working hypothesis.

1. The coefficients for the downward and upward formulas are the same in absolute value for corresponding terms.
2. These coefficients for class m, degree d and shift k are as follows:

$$F[m,\ d+k,\ d]F[m,\ k-1,\ 0]$$
$$F[m,\ d+k,\ d-1]F[m,\ k,\ 1]$$
$$F[m,\ d+k,\ d-2]F[m,\ k+1,\ 2]$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$F[m,\ d+k,\ 1]F[m,\ k+d-2,\ d-1]$$
$$F[m,\ d+k,\ 0]F[m,\ k+d-1,\ d]\ .$$

3. The sign patterns for the various cases are:

	Down	Up
Odd degree		
m odd	+ + - -	$(-1)^k$[+ - - +]
m even	+ - + -	+ - + -
Even degree		
m odd	+ + - -	+ + - -
m even	+ - + -	+ - + -

Formulas covering all cases are as follows:

m odd

$$(9)\quad H[m,d,n+k] = \sum_{i=1}^{d+1}(-1)^{[(i-1)/2]}F[m,k+d,\ d-i+1]F[m,k+i-2,\ i-1]H[m,d,\ n-i+1]\ .$$

(10a) for d odd

$$H[m,d,\ n-k] = (-1)^k \sum_{i=1}^{d+1}(-1)^{[i/2]}F[m,\ k+d,\ d-i+1]F[m,\ k+i-2,\ i-1]H[m,d,n+i-1]\ .$$

(10b) for d even

$$H[m,\ d,\ n-k] = \sum_{i=1}^{d+1}(-1)^{[(i-1)/2]}F[m,\ k+d,\ d-i+1]F[m,\ k+i-2,\ i-1]H[m,\ d,\ n+i-1]\ .$$

m even

$$(11)\quad H[m,d,\ n+k] = \sum_{i=1}^{d+1}(-1)^{i-1}F[m,\ k+d,\ d-i+1]F[m,\ k+i-2,\ i-1]H[m,\ d,\ n-i+1]\ .$$

(12) $$H[m,d,n-k] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m,k+d,d-i+1]F[m,k+i-2,i-1]H[m,d,n+i-1] .$$

PROOF OF THESE FORMULAS

The formulas are true for $k = 1$ since formulas (9) through (12) reduce to the respective formulas in (5) through (8). Assume then that these formulas are true for k. To go to $k + 1$, substitute for $H[m, d, n]$ either up or down as the case may be, using formulas (5) through (8). Consider first the case of downward shift. For m odd, the sign pattern for terms after substitution is:

$$+ - - + + - - +$$
$$+ + - - + + - -$$

where the second line represents the sign pattern for the substituted quantity $H[m, d, n]$. For m even, the sign pattern for terms after substitution is

$$- + - + - +$$
$$+ - + - + -$$

In either case the pattern varies modulo 4.

m odd

By (5),

$$H[m,d,n] = F[m,d+1,1]H[m,d,n-1] + F[m,d+1,2]H[m,d,n-2]$$
$$- F[M,d+1,3]H[m,d,n-3] - F[m,d+1,4]H[m,d,n-4] \cdots .$$

Substitution into (9) gives:

$$\begin{aligned} H[m,d,n+k] &= \{F[m,d+k,d-1]F[m,k,1] + F[m,d+1,1]F[m,d+k,d]\}H[m,d,n-1] \\ &+ \{-F[m,d+k,d-2]F[m,k+1,2] + F[m,d+1,2]F[m,d+k,d]\}H[m,d,n-2] \\ &+ \{-F[m,d+k,d-3]F[m,k+2,3] - F[m,d+1,3]F[m,d+k,d]\}H[m,d,n-3] \\ &+ \{F[m,d+k,d-4]F[m,k+3,4] - F[m,d+1,4]F[m,d+k,d]\}H[m,d,n-4] \\ &\quad \cdots \end{aligned}$$

COEFFICIENT OF $H[m, d, n - 4j - 1]$

This is given by:

$$F[m,d+k,d-4j-1]F[m,k+4j,4j+1] + F[m,d+1,4j+1]F[m,d+k,d]$$

$$= \frac{F_{m(d+k)}F_{m(d+k-1)}\cdots F_{m(k+4j+2)}}{F_mF_{2m}\cdots F_{m(d-4j-1)}} \frac{F_{m(k+4j)}F_{m(k+4j-1)}\cdots F_{mk}}{F_mF_{2m}\cdots F_{m(4j+1)}}$$

$$+ \frac{F_{m(d+1)}F_{md}\cdots F_{m(d-4j+1)}}{F_mF_{2m}\cdots F_{m(4j+1)}} \frac{F_{m(d+k)}F_{m(d+k-1)}\cdots F_{m(k+1)}}{F_mF_{2m}\cdots F_{dm}} =$$

$$= \frac{F_{m(d+k)}F_{m(d+k-1)}\cdots F_{m(k+4j+2)}}{F_mF_{2m}\cdots F_{m(d-4j-1)}} \quad \frac{F_{m(k+4j)}F_{m(k+4j-1)}\cdots F_{m(k+1)}}{F_mF_{2m}\cdots F_{m(4j+1)}}$$

$$\times \left\{F_{mk} + \frac{F_{m(d+1)}F_{m(k+4j+1)}}{F_{m(d-4j)}}\right\} .$$

The expression

$$F_{mk}F_{m(d-4j)} + F_{m(d+1)}F_{m(k+4j+1)}$$

can be modified using the relation

$$F_aF_b + F_{a+r}F_{b+r} = F_rF_{a+b+r} , \tag{13}$$

where r is odd, giving the result

$$F_{m(4j+1)}F_{m(d+k+1)} ,$$

which leads to the final value of the coefficient of $H[m, d, n - 4j - 1]$ after substitution $F[m, d + k + 1, d - 4j]\ F[m, k + 4j, 4j]$.

That this is the correct value is seen from the following considerations. The quantity is a coefficient in the expansion for $H[m, d, n + k]$. If the subscripts of all the H's are raised by 1 to give an expression for $H[m, d, n + k + 1]$, it is seen that this quantity is the coefficient of $H[m, d, n - 4j]$. By formula (9) with k replaced by $k + 1$ and $i = 4j + 1$, we obtain for the coefficient of $H[m, d, n - 4j]$ the value

$$F[m, k + d + 1, d - 4j - 1 + 1]F[m, k + 1 + 4j + 1 - 2, 4j + 1 - 1]$$
$$= F[m, k + d + 1, d - 4j]F[m, d + 4j, 4j]$$

the result obtained by our analysis.

COEFFICIENT OF $H[m, d, n - 4j - 2]$

The development proceeds as before, ending with the following quantity in brackets

$$-F_{mk} + \frac{F_{m(d+1)}F_{m(k+4j+2)}}{F_{m(d-4j-1)}} .$$

The expression

$$-F_{mk}F_{m(d-4j-1)} + F_{m(d+1)}F_{m(k+4j+2)} = F_{m(4j+2)}F_{m(d+k+1)}$$

using the relation

$$F_{a+c}F_{b+c} - F_aF_b = F_cF_{a+b+c} \quad \text{where } c \text{ is even} . \tag{14}$$

The final result is

$$F[m, d + k + 1, d - 4j - 1]F[m, k + 4j + 1, 4j + 1] .$$

This is the coefficient of $H[m, d, n - 4j - 2]$ in the expansion of $H[m, d, n + k]$. Shifting the H subscripts up one, this should be the coefficient of $H[m, d, n - 4j - 1]$ in the expansion of $H[m, d, n + k + 1]$. Replacing k by $k + 1$ and i by $4j + 2$ in (9) one finds for the coefficient of $H[m, d, n - 4j - 1]$ in the expansion of $H[m, d, n + k + 1]$

$$F[m, d + k + 1, d - 4j - 2 + 1]F[m, k + 1 + 4j + 2 - 2, 4j + 2 - 1]$$
$$= F[m, d + k + 1, d - 4j - 1]F[m, k + 4j + 1, 4j + 1]$$

in agreement with our analysis.

For the coefficient of $H[m, d, n - 4j - 3]$, the signs of the quantities in brackets are $-$ and $-$ with an odd subscript difference so that the formula (13) applies giving a result with a negative sign. For the coefficient of $H[m, d, n - 4j - 4]$ the signs are $+$ and $-$ with the latter predominating to give a negative sign.

The final term in the expansion after substitution would be

$$(-1)^{[d/2]}F[m, d + k, d]$$

This is to be compared with the final term in the expansion of $H[m, d, n + k + 1]$. Using formula (9) this should be

$$(-1)^{[(d+1-1)/2]}F[m, d + k + 1, d - d - 1 + 1]F[m, k + 1 + d + 1 - 2, d + 1 - 1]$$

or

$$(-1)^{[d/2]} F[m, d + k, d] \quad ,$$

which agrees with the result obtained by the substitution.

m even. Downward shift

The coefficients are the same so that we need simply consider the sign pattern.

	4j + 1	4j + 2	4j + 3	4j + 4
(1)	-	+	-	+
(2)	+	-	+	-
(3)	+	-	+	-

(1) is the sign pattern of terms that remain in the original expression.

(2) is the sign pattern of the substituted terms.

(3) is the sign pattern of the resulting expression.

Since the subscript differences are all even, formula (14) will apply in all these cases.

m odd, d odd, k odd, Upward shift

	4j + 1	4j + 2	4j + 3	4j + 4
(1)	+	+	-	-
(2)	+	-	-	+
(3)	+	-	-	+

Not e that for $4j + 1$ and $4j + 3$ the subscript difference is odd and formula (13) applies. The final set of signs is correct since $k + 1$ is even.

m odd, d odd, k even. Upward shift

	4j + 1	4j + 2	4j + 3	4j + 4
(1)	-	-	+	+
(2)	-	+	+	-
(3)	-	+	+	-

The final set of signs is correct since $k + 1$ is odd.

m odd, d even, Upward shift

	4j + 1	4j + 2	4j + 3	4j + 4
(1)	+	-	-	+
(2)	+	+	-	-
(3)	+	+	-	-

the signs being correct for this case also.

For m even and upward shift, the sign pattern is the same as for m even and downward shift.

To conclude, two examples are given to show the generality of these shift formulas and the manner of using them.

Example 1. Let

$$H[3,4,n] = L^2_{3n-4} F_{3n+2} T_{3n-1}$$

We wish to express $H[3, 4, n+4]$ in terms of $H[3, 4, n]$ and quantities with lower subscript. By our shift formulas we have:

$$\begin{aligned} H[3, 4, n+4] = {} & F[3,8,4]H[3,4,n] + F[3,8,3]F[3,4,1]H[3, 4, n-1] \\ & - F[3,8,2]F[3,5,2]H[3,4, n-2] - F[3,8,1]F[3,6,3]H[3,4, n-3] \\ & + F[3,8,0]F[3,7,4]H[3, 4, n-4] \ . \end{aligned}$$

Let $n = 3$ for purposes of checking. Then

$$\begin{aligned} L^2_{17} F_{23} T_{20} = {} & 10212563270\, L^2_5 F_{11} T_8 + 2410834608 \cdot 72\, L^2_2 F_8 T_5 \\ & - 31721508 \cdot 5490\, L^2_{-1} F_5 T_2 - 23184\ 417240\, L^2_{-4} F_2 T_{-1} \\ & + 31716035\, L^2_{-7} F_{-1} T_{-4} \end{aligned}$$

or

7055823593395596

should equal

$$10212563270 \cdot 646140 + 173580091776 \cdot 2646 - 174151078920 \cdot 20$$
$$- 9673292160(-98) + 31716035 \cdot 10092 \quad ,$$

which checks out.

Example 2. Let

$$H[3,\ 4,\ n] = T^4_{3n-7}$$
$$H[3,\ 4,\ n-3] = F[3,7,4]H[3,4,n] + F[3,7,3]F[3,3,1]H[3,\ 4,\ n+1]$$
$$- F[3,7,2]F[3,4,2]H[3,4,\ n+2] - F[3,7,1]F[3,5,3]H[3,4,\ n+3]$$
$$+ F[3,6,4]H[3,\ 6,\ n+4] \ .$$

Let $n = 2$.

$$T^4_{-10} = 31716035\,T^4_{-1} + 31716035 \cdot 17\,T^4_2 - 1767779 \cdot 306\,T^4_5$$
$$- 5473 \cdot 5490\,T^4_8 + 98515\,T^4_{11}$$
$$T^4_{-10} = 212^4 = 2019963136 \ .$$

The right-hand side equals:

$$31716035 \cdot 16 + 539172595 \cdot 256 - 540940374 \cdot 38416 - 30046770 \cdot 12960000 + 98515 \cdot 4162314256$$

which checks.

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