THE GOLDEN RATIO AND A GREEK CRISIS*

G. D. (Don) CHAKERIAN
University of California, Davis, California

The story of the discovery of irrational numbers by the school of Pythagoras around 500 B. C., and the devastating effect of that discovery on the Pythagorean philosophy is well known. On the one hand there was an undermining of the Pythagorean dictum "All is number," the conviction that everything in our world is expressible in terms of integers or ratios of integers. On the other hand, many geometric arguments were invalidated. Namely, those proofs requiring the existence of a common unit of measurement for any given pair of line segments were seen to be incomplete. Credit for the discovery of incommensurables is generally accorded to Hippasus of Metapontum. One may read, for example, in the excellent treatise of Van der Waerden [5], the legends of the fate that befell Hippasus for publicizing this and other secrets of the Pythagoreans. A brief and very readable account of these matters may be found in Meschkowski [4].

This note concerns itself with the question of how incommensurables might plausibly have been discovered. In particular, it will be seen how a study of the Golden Ratio could lead one to stumble onto the existence of incommensurable segments. The basic idea presented here is certainly not new and represents only a slight variant of ideas suggested in Meschkowski [4] and a definitive article by Heller [2]. It is hoped that the presentation given here might be of pedagogical value. In particular, a development along the lines given here might serve as a suitable vehicle for a classroom investigation of topics dealing with the history of irrational numbers or topics involving early Greek geometry and the Golden Ratio.

We begin by recalling that two line segments are commensurable, that is, have a common unit of measure, if each can be subdivided into smaller segments of equal length u (the length u being the same for both segments). In this case, if the two given segments have lengths a and b, respectively, we have

$$a = mu \qquad \text{and} \qquad b = nu \tag{1}$$

for some positive integers m and n. Thus, for commensurable segments, we have the ratio $a/b = m/n$ is a rational number. Conversely, if we are given two line segments of lengths a and b such that the ratio a/b is equal to m/n, where m and n are positive integers, then the number $u \equiv a/m = b/n$ will serve as a common unit of measure, so the segments are commensurable. Thus commensurable pairs of line segments are precisely those for

*Revised version of a lecture given before the Fibonacci Association in San Francisco on April 22, 1972.

which the ratio of the lengths is a rational number, and incommensurable pairs are those for which the ratio is an irrational number.

The best known example of an incommensurable pair of segments is given by a side and diagonal of a square. In a square, the ratio of diagonal length to side length is $\sqrt{2}$, which an easy number theoretic argument (as given in Book X of Euclid's Elements) shows to be irrational. But historical evidence indicates that the discovery of incommensurables came about in a purely geometric fashion, and the known geometric proofs that diagonal and side of a square are incommensurable seem to have the nature of being concocted after the initial discovery was well known. The reader will find the standard geometric argument in Eves [1, p. 60]. One would like to see a pair of line segments whose incommensurability can be more intuitively grapsed in a purely geometric manner. This is where the Golden Ratio engers the scene.

The Pythagoreans were much taken with the properties of the regular pentagon, whose vertices are also the vertices of the Pythagorean symbol of health, the regular five-pointed star.

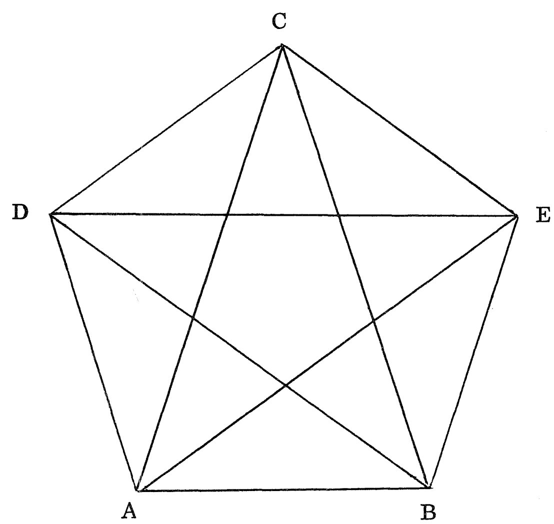

Figure 1

The Golden Ratio is the ratio of the diagonal length of a regular pentagon to the side length. Designating this ratio by the symbol ϕ, we have from Fig. 1,

$$\phi = \frac{AC}{AB} . \tag{2}$$

Some simple geometry shows that in Fig. 1, triangle ACB is an isosceles triangle with apex angle 36° and base angles 72° each. Such a triangle we shall call a Golden Triangle. Then the Golden Ratio is the ratio of side to base in any Golden Triangle. A property of the Golden

Triangle that undoubtedly intrigued the Pythagoreans is that when one draws the bisector of a base angle, there appears another smaller Golden Triangle. Thus in Fig. 2, if triangle ACB is a Golden Triangle and AD bisects the angle at A, then triangle BAD is also a Golden Triangle.

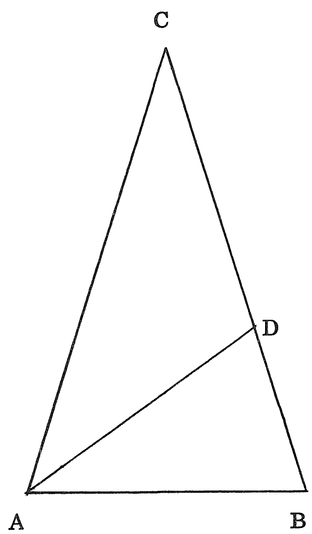

Figure 2

To see why this is true, observe in Fig. 2, that $\measuredangle$ BAD = $\measuredangle$ CAD = 36° and $\measuredangle$ ABD = 72°. It follows that $\measuredangle$ ADB = 72°, so triangle BAD is indeed a Golden Triangle. In this self-replicating property of the Golden Triangle lies the key to the incommensurability of its side and base. If one next draws the bisector of the angle at D to a point D' on AB, then draws the bisector of the angle at D' to a point D'' on BD, and continues this process indefinitely, one obtains an infinite sequence of smaller and smaller Golden Triangles. We shall see in a moment how the existence of this sequence contradicts the possibility that the side and base of the triangle might be commensurable.

It will be crucial to our argument to observe that in Fig. 2, AD = CD, which follows from the fact that $\measuredangle$ DAC = $\measuredangle$ DCA = 36°.

How then does one see geometrically that the side and base of a Golden Triangle are not commensurable? We might place ourselves in the sandals of an ancient Greek philosopher ruminating over a Golden Triangle ACB sketched in the sand. Wondering about a common unit of measure of AC and AB, we imagine it is possible to subdivide AC and AB into smaller segments all of the same length, say u. Subdividing BC into segments of the same length u we obtain an "evenly subdivided" triangle that might look something like triangle ACB in Fig. 3, where all the little segments are supposed to have the same length u. Now comes a crucial observation. Suppose we draw the bisector of the base angle at A, intersecting the opposite side in a point D. What can we say about D? The crucial observation

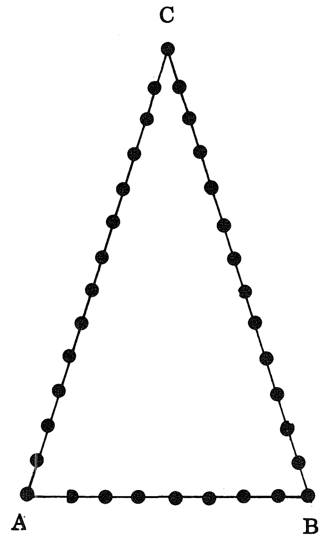

Figure 3

is that D must be one of the subdivision points! The reason is simple. Referring to Fig. 2, recall that AB = AD = CD. Thus CD, being equal to AB, must be an integral multiple of u, hence D must be a subdivision point. Thus appears a basic revelation: <u>If we have any evenly subdivided Golden Triangle, then the bisector of a base angle must strike the opposite side in a subdivision point.</u> Figure 4 illustrates this, with the bisector AD also subdivided.

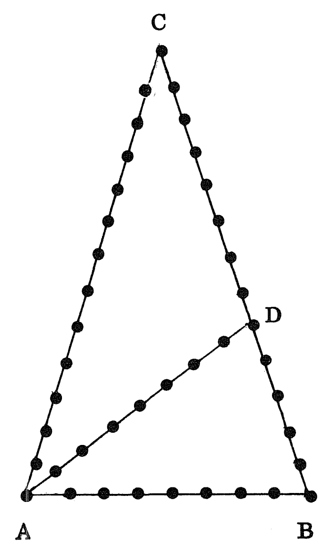

Figure 4

But triangle BAD is also an evenly subdivided Golden Triangle; hence if the bisector DD' of $\measuredangle$ ADB is drawn, the point D' where it strikes side AB must also be one of the original subdivision points, as indicated in Fig. 5.

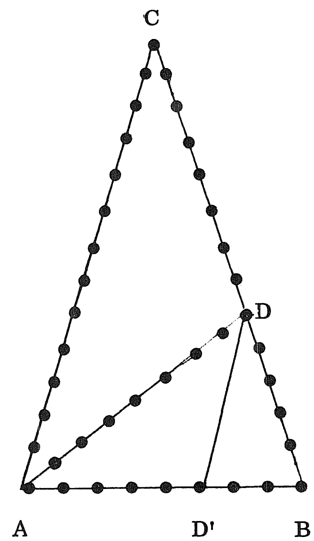

Figure 5

Repeating the process on the evenly subdivided Golden Triangle D'DB, we next see that the bisector of $\measuredangle$ BD'D must strike BD in one of the original subdivision points D''. It now becomes clear that we can repeat this procedure endlessly, drawing successively angle bisectors DD', D'D'', D''D''', $\cdots$, striking at each step the different subdivision points D', D'', D''', $\cdots$. Since at each step of this procedure we strike one of our original subdivision points, we have arrived at a contradiction, there being only finitely many such points. Thus we see that an evenly subdivided Golden Triangle is impossible, and hence the side and base are not commensurable.

It is of interest to examine an algebraic proof of the irrationality of the Golden Ratio ϕ that parallels the preceding geometric argument. We begin by deriving an important equation satisfied by ϕ. Since ACB and BAD are similar Golden Triangles in Fig. 2, we have

$$\phi = \frac{BC}{AB} = \frac{AB}{BD} = \frac{AB}{BC - DC} = \frac{AB}{BC - AB} = \frac{1}{\phi - 1} \quad , \tag{3}$$

where we also used the fact that DC = AB and made some minor algebraic adjustments. If now we have $\phi = m/n$, with m, n positive integers, then Eq. (3) implies

$$\frac{m}{n} = \phi = \frac{1}{\phi - 1} = \frac{n}{m - n} \quad . \tag{4}$$

Since $\phi > 1$, we automatically have $m > n$, and defining $m' = n$ and $n' = m - n$, we obtain $\phi = m'/n'$, with m', n' positive integers and $m > m'$. Repeating the process we obtain positive integers m'', n'' with $\phi = m''/n''$ and $m' > m''$. Repeating the procedure endlessly we obtain an infinite decreasing sequence of positive integers $m > m' > m'' > \cdots$, a contradiction. Hence there do not exist positive integers m, n such that $\phi = m/n$, and we have proved that ϕ is not rational.

In both the preceding proofs we may avoid the construction of infinite sequences by appealing to the fact that any nonempty set of positive integers contains a smallest element. In the case of our geometric proof, suppose there existed evenly subdivided Golden Triangles. With each such subdivided triangle associate the total number of subdivision points. Then there is a smallest such integer N and corresponding evenly subdivided Golden Triangle. But then by bisecting a base angle of this triangle we produce an evenly subdivided Golden Triangle with less than N total subdivision points. This contradiction shows that there exist no evenly subdivided Golden Triangles. In the case of our algebraic proof of the irrationality of ϕ, suppose there existed positive integers m and n such that $\phi = m/n$. With each such representation associate the numerator m. Then there is a smallest such integer m for which $\phi = m/n$. But then Eq. (4) gives $\phi = n/(m - n)$, which is a representation with still smaller numerator. The contradiction shows that there is no representation $\phi = m/n$ with positive integers m and n. Hence ϕ is not rational.

No discussion of these matters would be complete without mentioning how from Eq. (3), or from its equivalent

$$\phi = 1 + \frac{1}{\phi}, \tag{5}$$

one may obtain rational approximations to ϕ by ratios of successive Fibonacci numbers, with the analogous geometric approximations to a Golden Triangle by integer-sided triangles. Having mentioned it, we now leave it, hoping that any reader unfamiliar with these matters will, with whetted appetite, consult the fine book of Hoggatt [3] for a detailed exposition of the relationship between geometry and the Fibonacci numbers.

REFERENCES

1. Howard Eves, An Introduction to the History of Mathematics, Holt, Rinehart and Winston, New York, 1969.
2. S. Heller, Die Entdeckung der stetigen Teilung durch die Pythagoreer, Abh. Ak. der Wiss., Klasse f. Math., Phys., u. Technik, No. 6, v. 54 (1958).
3. V. E. Hoggatt, Fibonacci and Lucas Numbers, Houghton Mifflin, Boston, 1969.
4. H. Meschkowski, Ways of Thought of Great Mathematicians, Holden-Day, San Francisco, 1964.
5. B. L. Van der Waerden, Science Awakening, Oxford University Press, New York, 1961.