A RELIABLITY PROBLEM

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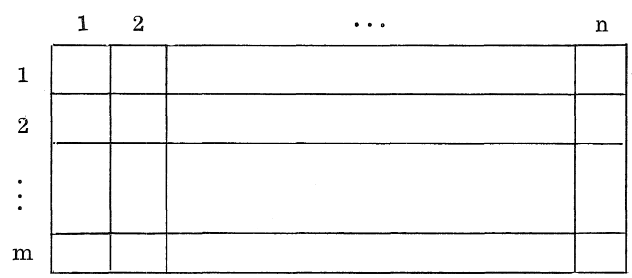
ABSTRACT

An $m \times n$ array of elements is considered in which each element has a probability p of being reliable. The array as a whole is considered reliable if there does not exist in the array any polydominoe of a given form in any orientation having all of its elements unreliable. A method is given for determining the probability of reliability for the array and solutions are worked out explicitly for several special cases.

1. INTRODUCTION

We are given an $m \times n$ array



in which each of the mn elements has a given probability p of being reliable (and a probability q of being unreliable where $p + q = 1$). The $m \times n$ matrix as a whole will be considered reliable, if and only if, there does not exist in the array, any polydominoe of a given form in any orientation having all of its elements unreliable. The problem then is to calculate the probability of reliability of the array. The special case where $m = 2$ and the given polydominoe is a 2×1 arose in the design of a low-altitude detection antenna. If any 2×1 polydominoe had both its elements unreliable, then the antenna could not fulfill its detection mission.

The specific cases to be considered here explicitly are the following:

	Array size	Failure polydominoe
(C_1)	$2 \times n$	q | q

	Array size	Failure polydominoe
(C_2)	$2 \times n$	q q q
(C_3)	$2 \times n$	q q q q
(C_4)	$3 \times n$	q q
(C_5)	$3 \times n$	q q q q

2. PROBABILITY FOR RELIABILITY

For all the cases, we will let P_n denote the probability of the $2 \times n$ or $3 \times n$ array being reliable. For the $2 \times n$ array, A_n, B_n, C_n, D_n will denote the respective probabilities of reliability of the array if the end 2×1 polydominoe has the form

p	p	q	q
p	q	p	q

and then

(1)
$$P_n = A_n + B_n + C_n + D_n .$$

For the case (C_1), $D_n = 0$ and $B_n = C_n$. Here, for an A_{n+1} array, the end 2×2 polydominoe must have one of the three following forms:

p	p	p	p	q	p
p	p	q	p	p	p

Thus,

(2)
$$A_{n+1} = p^2(A_n + B_n + C_n) .$$

For a B_{n+1} array, the end 2×2 polydominoe must have one of the two forms

p	p	q	p
p	q	p	q

and thus

(3)
$$B_{n+1} = pq(A_n + C_n)$$

and similarly

(4)
$$C_{n+1} = pq(A_n + B_n) .$$

On eliminating B_n and C_n, we obtain

(5) $$A_{n+1} = p^2P_n ,$$

(6) $$P_{n+1} = pP_n + pqA_n ,$$

and then

(7) $$P_{n+1} = pP_n + p^3qP_{n-1} .$$

For initial conditions, we have

(8) $$A_1 = p^2, \quad B_1 = pq = C_1, \quad D_1 = 0.$$

Whence,

(9) $$P_1 = 1 - q^2, \quad P_2 = 2p^2 - p^4 .$$

The solution of (7) is then given by

$$P_n = k_1r_1^n + k_2r_2^n ,$$

where r_1, r_2 are the roots of $x^2 = px + p^3q$ and constants k_1, k_2 are determined so as to satisfy (9). This gives

(10) $$P_n = \frac{1-q^2}{a}\left\{\left(\frac{p+a}{2}\right)^n - \left(\frac{p-a}{2}\right)^n\right\} + \frac{p^3q}{a}\left\{\left(\frac{p+a}{2}\right)^{n-1} - \left(\frac{p-a}{2}\right)^{n-1}\right\} ,$$

where $a = \sqrt{p^2 + 4p^3q}$.

For (C_2), it then follows as before that

(11) $$A_{n+1} = p^2(A_n + B_n + C_n + D_n) ,$$

(12) $$B_{n+1} = C_{n+1} = pq(A_n + B_n + C_n) ,$$

(13) $$D_{n+1} = q^2A_n$$

subject to the initial conditions,

(14) $$A_1 = p^2, \quad B_1 = pq = C_1, \quad D_1 = q^2 .$$

Eliminating B_n, C_n, D_n, we obtain

(15) $$A_{n+2} = (p^2 + 2pq)A_{n+1} + p^2q^2A_n - 2p^3q^3A_{n-1} .$$

Whence,

$$A_n = k_1r_1^n + k_2r_2^n + k_3r_3^n ,$$

where r_1, r_2, r_3 are the roots of

$$x^3 = (p^2 + 2pq)x^2 + p^2q^2x - 2p^3q^3$$

and the constants k_1, k_2, k_3 are determined from the initial conditions (note that here $A_1 = A_2 = p^2$, $A_3 = p^4[1 + 2pq + 5q^2]$). Then B_n, C_n, D_n and P_n are easily determined.

For (C_3), we have (11) and

(16) $$B_{n+1} = C_{n+1} = pq(A_n + B_n + C_n + D_n) ,$$

(17) $$D_{n+1} = q^2(A_n + B_n + C_n)$$

(again all subject to conditions (14)). On eliminating D_n, we obtain

(18) $$A_{n+1} = p^2\ A_n + B_n + C_n + q^2(A_{n-1} + B_{n-1} + C_{n-1}) ,$$

(19) $$pB_{n+1} = pC_{n+1} = qA_{n+1} .$$

Whence,

(20) $$A_{n+1} = p(p + 2q)(A_n + q^2A_{n-1}) .$$

Then,

$$A_n = k_1r_1^n + k_2r_2^n ,$$

where r_i are the roots of

$$x^2 = p(p + 2q)(x + q3)$$

and the k_i's are determined from the initial conditions.

Then B_n, C_n, D_n and P_n are found from (14), (17) and (1).

For the $3 \times n$ arrays, we let A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n denote the respective probabilities of reliability of the array if the end 3×1 polydominoe has the form

$$\begin{array}{|c|}\hline p\\\hline p\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline p\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline q\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline p\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline q\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline p\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline q\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline q\\\hline q\\\hline\end{array}$$

and

(21) $$P_n = A_n + B_n + C_n + D_n + E_n + F_n + G_n + H_n .$$

For (C_4),

$$E_n = G_n = H_n = 0, \quad B_n = D_n ,$$

(22) $$A_1 = p^3, \quad B_1 = C_1 = D_1 = p^2q, \quad F_1 = pq^2 ,$$

(23) $$A_{n+1} = p^3(A_n + B_n + C_n + D_n + F_n) ,$$

(24) $$B_{n+1} = D_{n+1} = p^2q(A_n + C_n + D_n) ,$$

(25) $$C_{n+1} = p^2q(A_n + B_n + D_n + F_n) ,$$

(26) $$F_{n+1} = pq^2(A_n + C_n) .$$

For (C_5),

$$B_n = C_n = D_n, \quad E_n = G_n,$$

(27) $$A_1 = p^3, \quad B_1 = C_1 = D_1 = p^2q, \quad E_1 = F_1 = G_1 = pq^2, \quad H_1 = q^3 ,$$

$$A_{n+1} = p^3P_n \ , \tag{28}$$

$$B_{n+1} = p^2qP_n \ , \tag{29}$$

$$E_{n+1} = pq^2(A_n + B_n + C_n + D_n + F_n + G_n) \ , \tag{30}$$

$$F_{n+1} = pq^2P_n \ , \tag{31}$$

$$H_{n+1} = q^3(A_n + B_n + C_n + D_n + F_n) \ . \tag{32}$$

Although we can carry out the elimination process for (C_4) and (C_5) by means of the operator E and then determine P_n in terms of the roots of a higher order polynomial, it is not worthwhile. In these cases (and even some of the prior ones), one can just use a computer on the recurrence relations to determine the P_n's.

3. HIGHER ORDER POLYDOMINOES

The previous methods, with some adaptation, will also apply when the failure polydominoe is of higher order than the previous ones. As in the last two cases, it will suffice to just get the appropriate recurrence equations. If the failure polydominoe is of the type

q	q	q

in a $3 \times n$ array, then we would need terms A_n, B_n, $\cdots$, corresponding to a reliable $3 \times n$ array whose end 2×3 polydominoe has the forms

p	p
p	p
p	p

,

p	p
p	p
p	q

,

p	p
p	q
p	p

etc.

This will, of course, lead to an increased number of recurrence relations. Other arrays which can be solved similarly are cylindrical and torodial ones as well as higher dimensional ones.