SOME CORRECTIONS TO CARLSON'S "DETERMINATION OF HERONIAN TRIANGLES"

DAVID SINGMASTER
Polytechnic of the South Bank, London, SE1

In [1], Carlson presents a determination of all Heronian triangles, i.e., triangles with integral sides and area. He correctly shows that every such triangle, or a multiple thereof, can be split into two Pythagorean triangles, i.e., right triangles with integer sides. Unfortunately, he then makes a common error in incorrectly assuming that all Pythagoreal triangles are of the form:

(1) $$u^2 + v^2, \quad u^2 - v^2, \quad 2uv,$$

rather than the correct form:

(2) $$a(u^2 + v^2), \quad a(u^2 - v^2), \quad 2auv,$$

(One can easily verify that 15, 9, 12 is a Pythagorean triangle which cannot be expressed by (1). The same error is also made in [2].)

Using the form (2) with Carlson's main theorem, we have the following correct form of his

Corollary 1. A triangle is Heronian if and only if its sides can be represented as

(i) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad a(u^2 - v^2) + b(r^2 - s^2),$$

where $auv = brs$:

(ii) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad a(u^2 - v^2) + 2brs,$$

where $2auv = b(r^2 - s^2)$:

(iii) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad 2auv + 2brs,$$

where $a(u^2 - v^2) = b(r^2 - s^2)$; or:

(iv) a reduction by a common factor of a triangle given by (i), (ii), or (iii).

Carlson's incorrect form of Corollary 1 had three conditions numbered (3), (4) and (5) corresponding to our (i), (iii) and (iv) without the parameters a and b. It appears that the original Corollary 1 neglected to consider that the common side of the two Pythagorean triangles might be of the form $u^2 - v^2$ in one and of the form $2rs$ in the other. If one constructs a short table of Pythagorean triples from (1), one has:

u	v	$u^2 + v^2$	$u^2 - v^2$	2uv
2	1	5	3	4
3	1	10	8	6
3	2	13	5	12
4	1	17	15	8
4	2	20	12	16
4	3	25	7	24

One immediately wants to construct a Heronian triangle from 10, 8, 6 and 17, 15, 8, obtaining 10, 17, 21. This construction is of form (ii) of the corrected Corollary 1, but does not fit into Carlson's (3), (4) or (5). To see that (5) does not apply, suppose that it did. Then we must put together either:

10c, 6c, 8c and 17c, 15c, 8c, which gives $16c = 2u^2$ and $32c = 2r^2$, which is impossible; or:

10c, 8c, 6c and 17c, 8c, 15c, which gives $18c = 2u^2$ and $25c = 2r^2$, which is impossible.

(Possibly Corollary 1 may apply in its original form to the splitting by one of the other altitudes.)

Carlson's Lemma 2 is unclear. (The following remarks assume the reader has 1 at hand.) What he has proven, but not clearly stated, is that an isosceles Heronian triangle is obtained by putting together two equal Pythagorean triangles. The first step of the proof, that a primitive Heronian triangle has only one even side, is not proven until four pages later, on p. 505. The fact that the side of the isosceles triangle is odd is not used, only the fact that the base is even is needed and this holds for any isosceles Heronian triangle since it holds for primitive ones. Carlson's parameter Q is simply the altitude on the base and his result $A = nQ$ is direct from the ordinary area formula. (There may be some historical interest in using Hero's formula for the area, but I would not consider the added interest to be worth the added complexity.) Further, to obtain primitiveness, one must make assumptions on GCD (u,v) and the parity of u and v. We give the following clearer and correct form of Carlson's

Lemma 2. A triangle is an isosceles Heronian triangle if and only if its sides can be represented as:

(i) $$a(u^2 + v^2), \quad a(u^2 + v^2), \quad 4auv;$$

or:

(ii) $$a(u^2 + v^2), \quad a(u^2 + v^2), \quad 2a(u^2 - v^2).$$

The triangle is then primitive if and only if

$$a = 1, \quad \mathrm{GCD}(u,v) = 1 \quad \text{and} \quad u \not\equiv v \pmod 2).$$

Incidentally, it is possible to obtain different representations of isosceles Heronian triangles. Consider the Pythagorean triangles 30, 18, 24 (obtained from $u = 2$, $v = 1$, $a = 6$) and 25, 7, 24 (obtained from $u = 4$, $v = 3$, $a = 1$). These fit together to form the isosceles Heronian triangle 25, 25, 30 which reduces to 5, 5, 6.

REFERENCES

1. John R. Carlson, "Determination of Heronian Triangles," Fibonacci Quarterly, Vol. 8 (1970), pp. 499-506 and 551.
2. W. J. LeVeque, "A Brief Survey of Diophantine Equations," Studies in Number Theory, W. J. LeVeque, Ed., Mathematical Ass'n. of America, 1969.