ON K-NUMBERS

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1. INTRODUCTION

We call K-numbers the numbers defined by

$$(1) \qquad K(k,n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} m^k .$$

In [1, p. 249], the following results are given: $K(k,n) = 0$ for $k < n$ and $K(1,1) = (-1)^k k!$. We shall study general K-numbers and shall complete the definition by writing $K(k,n) = 0$ for $k,n \le 0$. We shall use two results given in [1]:

$$(2) \qquad \sum_{s=0}^{t} \binom{s}{p} = \binom{t+1}{p+1}$$

cf. p. 246, No. 3, and

$$(3) \qquad t^\alpha - \binom{p}{1}(t+1)^\alpha + \binom{p}{2}(t+2)^\alpha + \cdots + (-1)^p \binom{p}{p}(t+p)^\alpha = \sum_{q=0}^{p} (-1)^q (t+q)^\alpha \binom{p}{q} = 0 ,$$

cf. p. 249.

The K-numbers are met in certain problems in combinatorics.

2. RECURRENCE RELATION

It will be observed in (1) that the term for $m = 1$ can be omitted since it is zero. Consider

$$K(k,n+1) = \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} m^k$$

and the difference

$$S = K(k,n+1) - K(k,n) = \sum_{m=0}^{n+1} (-1)^m \left[\binom{n+1}{m} - \binom{n}{m}\right] = \sum_{m=0}^{n+1} (-1)^m \binom{n}{m-1} m^k,$$

where use has been made of the relation

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}.$$

But

$$\binom{a}{b} = \frac{b+1}{a+1}\binom{a+1}{b+1},$$

so that

$$S = \frac{1}{n+1}\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}m^{k+1} = K(k+1,\ n+1)/(n+1),$$

thus the K-numbers satisfy the recurrence relation,

(4) $$K(k+1,\ n+1) = (n+1)[K(k,\ n+1) - K(k,n)]$$

or

(4a) $$K(k,n) = n[K(k-1,\ n) - K(k-1,\ n-1)].$$

3. NUMERICAL RESULTS

We observe that

(5) $$K(k,1) = \sum_{m=0}^{1}(-1)^m\binom{1}{m}m^k = -1.$$

Using the results of Section 1 and (4), we obtain the following table of $K(k,n)$:

k \ n	1	2	3	4	5	6	7
1	-1						
2	-1	2					
3	-1	6	-6				
4	-1	14	-36	24			
5	-1	30	-150	240	-120		
6	-1	62	-540	1560	-1800	720	
7	-1	126	-1806	8400	-16800	15120	-5040

4. HORIZONTAL SUMS

Consider the "horizontal sum"

$$S = \sum_{n=0}^{k} K(k,n) = \sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^m\binom{n}{m}m^k = \sum_{m=0}^{k}(-1)^m m^k\sum_{n=0}^{k}\binom{n}{m},$$

and using (2)

$$S = \sum_{m=0}^{k} (-1)^m m^k \binom{k+1}{m+1}.$$

Let in (3)

$$q = m + 1, \quad t = -1, \quad \alpha = k, \quad p = k + 1,$$

then (3) becomes

$$\sum_{m=-1}^{k+1} (-1)^{m+1} m^k \binom{k+1}{m+1} = 0,$$

or

$$(-1)^k - \sum_{m=0}^{k} (-1)^m m^k \binom{k+1}{m+1} = 0,$$

thus

$$S = \sum_{n=0}^{k} K(k,n) = (-1)^k. \tag{6}$$

5. GENERATING FUNCTION FOR THE K-NUMBERS

To find the generating function of the K-numbers, we use a technique given in [2]. We have

$$GK(k,n) = \sum_{k=0}^{\infty} K(k,n)t^k = u(n,t), \tag{7}$$

and

$$\sum_{k=n-1}^{\infty} K(k+1, n)t^k = GK(k,n)/t = u(n,t)/t$$

$$\sum_{k=n}^{\infty} K(k+1, n+1)t^k = GK(k, n+1)/t = u(n+1, t)/t = GK(k+1, n+1).$$

According to (4) it follows that

$$GK(k+1, n+1) = (n+1)[GK(k, n+1) - GK(k,n)]$$

or, substituting,

$$u(n+1, t)/t = (n+1)u(n+1, t) - (n+1)u(n,t),$$

which shows that $u(n,t)$ is a solution of the difference equation

(8) $$[1 - t(n+1)]u(n+1, t) + t(n+1)u(n,t) = 0 .$$

We solve (8) using the classical technique given in [2] and obtain

$$u(n,t) = \frac{(2t)(3t)\cdots(nt)}{(2t-1)(3t-1)\cdots(nt-1)}\,u(1,t) = n!\,\Gamma\left(2 - \frac{1}{t}\right)u(1,t)\Big/\Gamma\left(n + 1 - \frac{1}{t}\right).$$

According to (5), $K(k,1) = -1$, thus

$$u(1,t) = \sum_{k=1}^{\infty} K(k,1)t^k = -\sum_{k=1}^{\infty} t^k = t/(t-1), \qquad |t| < 1,$$

thus substituting into $u(n,t)$

(9) $$u(n,t) = GK(k,n) = n!\,\Gamma\left(1 - \frac{1}{t}\right)\Big/\Gamma\left(n + 1 - \frac{1}{t}\right),$$

which is the generating function for the K-numbers.

6. QUASI-ORTHOGONAL NUMBERS OF THE K-NUMBERS

According to [3] and correcting an error committed there, since the K-numbers satisfy a relation of the form

$$B_k^n = \frac{M(n+1)}{N(n+1)} B_{k-1}^n + \frac{1}{N(n)} B_{k-1}^{n-1},$$

where clearly (cf. (4a)), $N(n) = -1/n$, $M(n) = (n-1)/n$, so that, still according to [3] the quasi-orthogonal numbers satisfy the relation

$$A_k^n = M(k)A_{k-1}^n + N(k)A_{k-1}^{n-1};$$

calling $L(k,n)$ the numbers quasi-orthogonal to the numbers $K(k,n)$, we have

(10) $$L(k,n) = \frac{(k-1)}{k} L(k-1, n) - \frac{1}{k} L(k-1, n-1).$$

Through the quasi-orthogonality condition we get $L(k,k) = (-1)^k/k!$, and since $K(k,1) = -1$ it follows that for $k > 1$,

$$\sum_{n=1}^{k} K(k,n) = 0 .$$

It will also be easily verified that $L(k,1) = -1/k$. We thus obtain the following table of values of $L(k,n)$:

n:	1	2	3	4	5
k=					
1	-1				
2	-1/2	1/2			
3	-1/3	1/2	-1/6		
4	-1/4	11/24	-1/4	1/24	
5	-1/5	5/12	-7/24	1/12	-1/120

7. RELATIONS TO STIRLING NUMBERS

We consider the numbers

$$\omega(k,n) = (-1)^k k!\, L(k,n) \quad ,$$

i.e.,

$$L(k,n) = (-1)^k \omega(k,n)/k! \quad .$$

By substituting into (10) we obtain

$$\omega(k,n) = -(k-1)\omega(k-1,\, n) + \omega(k-1,\, n-1) \; ,$$

which is the recurrence relation for Stirling numbers of the first kind (cf. [2, p. 143]). Since $\omega(1,1) = 1 = St(1,1)$, $\omega(2,1) = -1 = St(2,1)$, etc., it follows that $\omega(k,n) = St(k,n)$, the Stirling numbers of the first kind, thus

(12) $$L(k,n) = (-1)^k St(k,n)/k! \quad .$$

Similarly it can be easily checked that the K-numbers are related to the Stirling numbers of the second kind $st(k,n)$ by the relation

(12a) $$K(k,n) = (-1)^n n!\, st(k,n) \quad .$$

REFERENCES

1. E. Netto, Lehrbuch der Kombinatorik, Chelsea, N. Y., Reprint of the second edition of 1927.
2. Ch. Jordan, Calculus of Finite Differences, Chelsea, N. Y., 1950.
3. S. Tauber, "On Quasi-Orthogonal Numbers," Amer. Math. Monthly, 72 (1962), pp. 365-372.