SOME GENERALIZATIONS SUGGESTED BY GOULD'S SYSTEMATIC TREATMENT OF CERTAIN BINOMIAL IDENTITIES

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In a previous article [1], the writer has presented properties of certain numbers A_n defined by the generating function

$$(1)\qquad f(n) = (1-u)^{-1}(1+u)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n u^n \quad .$$

In addition, Professor H. W. Gould of West Virginia University, in a recently published paper [2], has indicated several additional identities for the A_n coefficients.

We now introduce the generalized numbers $A_n(x)$ defined by the generating function

$$(2)\qquad g(u,x) = (1-u)^{-1}(1+u)^{x} = \sum_{n=0}^{\infty} A_n(x)\, u^n \quad ,$$

which is valid for all real or complex x; from this, the following relations are evident:

$$(3)\qquad f(u) = g(u, -\tfrac{1}{2})$$

$$(4)\qquad A_n = A_n(-\tfrac{1}{2})$$

$$(5)\qquad A_n(x) = \sum_{k=0}^{n} \binom{x}{k} \quad ,$$

where the combinatorial coefficients satisfy the basic relations:

$$(6)\qquad \binom{x}{k} = \frac{x^{(k)}}{k!} = \frac{x(x-1)\cdots(x-k+1)}{k!}\,; \qquad \binom{x}{0} = 1 \quad .$$

$$(7)\qquad \binom{-x}{k} = (-1)^k \binom{x+k-1}{k} \quad .$$

A useful special result is the identity:

$$\binom{-\frac{1}{2}}{k} = (-\tfrac{1}{4})^k \binom{2k}{k}$$

The purpose of this paper is to present some properties of the coefficients $A_n(x)$. Gould [2] has demonstrated that most of the identities shown in [1] are old results, citing numerous references to substantiate this claim. Likewise, in private communications with the writer, Gould has indicated that the coefficients $A_n(x)$ have been studied extensively by previous mathematicians. However, Gould [2] indicated that one identity proven in [1] appeared to be new in the literature, and restated it in the following form:

$$A_n^2 = \left\{\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\right\}^2 = \binom{-\frac{1}{2}}{n}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{n-k}\frac{2n+1}{2k+1} \qquad (8)$$

Gould, who as well as being a mathematician of the highest order, is an expert in the field of information retrieval, was impressed by the apparent novelty of the relation in (8), and his closing remarks in [2] stimulated a search for a suitable generalization of (8). This search was initiated by the writer in an effort to find a single-sum expression for the coefficient $A_n^2(x)$. In this respect, he has failed. However, the writer did discover an unexpected generalization of (8) by empirical methods, and this is expressed in the following elegant form:

$$\begin{aligned} A_n(x-\tfrac{1}{2})A_n(-x-\tfrac{1}{2}) &= \left\{\sum_{k=0}^{n}\binom{x-\frac{1}{2}}{k}\right\}\cdot\left\{\sum_{k=0}^{n}\binom{-x-\frac{1}{2}}{k}\right\} \\ &= \tfrac{1}{2}\binom{x-\frac{1}{2}}{n}\sum_{k=0}^{n}\binom{-x-\frac{1}{2}}{n-k}\frac{x-\frac{1}{2}-n}{x-\frac{1}{2}-k} + \tfrac{1}{2}\binom{-x-\frac{1}{2}}{n}\sum_{k=0}^{n}\binom{x-\frac{1}{2}}{n-k}\frac{x+\frac{1}{2}+n}{x+\frac{1}{2}+k} \end{aligned} \qquad (9)$$

It is easily seen that when $x = 0$, Eq. (9) reduces to (8). Relation (9) would appear to be a new combinatorial identity.

Before we furnish a proof of (9), we will present a list of various identities involving the $A_n(x)$ coefficients, each identity accompanied with a brief indication of the method used in its derivation. The purpose is to familiarize the reader with some of the known results. The $A_n(x)$'s satisfy the following second-order recursion:

$$(n+1)A_{n+1}(x) - (x+1)A_n(x) + (x-n)A_{n-1}(x) = 0 . \qquad (10)$$

Recursion (10) is easily verified from the definition of $A_n(x)$ in (5). For $x = -\frac{1}{2}$, it becomes recursion (7) in [1].

$$A_n(x) = (-1)^n\sum_{k=0}^{[n/2]}(-1)^k\binom{x}{k}\binom{-x-1}{n-2k} ; \qquad (11)$$

derived by expressing $g(u,x)$ in the form $(1-u)^{-1-x}(1-u^2)^x$, and obtaining the convolute.

For $x = \frac{1}{2}$, (11) becomes formula (11) in [2].

$$A_n(x) = (-1)^n \binom{x}{n} \sum_{k=0}^{n} (-2)^k \binom{n}{k} \frac{x-n}{x-k} \quad ; \tag{12}$$

derived by expressing $g(u,x)$ in the form

$$(1-u)^{x-1}\left(1 + \frac{2u}{1-u}\right)^x ,$$

and obtaining the convolute. For $x = -\frac{1}{2}$, this becomes formula (12) in [2], which was previously stated in variant form as formula (22) in [1]. NOTE: Identity (12) has been submitted to Advanced Problems Editor as a proposed problem.

$$A_n(x) = \binom{x}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k}} 2^k \frac{x-n}{x-k} \quad ; \tag{13}$$

derived by obtaining the convolute of the function $g(u,x)$ expressed in the form

$$(1+u)^{x-1}\left(1 - \frac{2u}{1+u}\right)^{-1} .$$

For $x = -\frac{1}{2}$, this becomes formula (13) in [2].

$$\begin{aligned} e^{\frac{1}{2}u^2} \int_0^u t^{-2x-1} e^{-t^2} dt &= \sum_{n=0}^{\infty} \frac{u^{2n}}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k-2x}}{k!\,(2k-2x)} \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n A_n(x) u^{2n-2x}}{(2n-2x)\binom{x}{n} n!} \quad ; \end{aligned} \tag{14}$$

(using (12) above). For $x = -\frac{1}{2}$, this becomes formula (11) in [1], restated by Gould as formula (14) in [2].

$$2^n = \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k}} A_k(x) \frac{x-n}{x-k} \tag{15}$$

$$(-2)^n = \binom{x}{n} \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{x}{k}} A_k(x) \frac{x-n}{x-k} \quad . \tag{16}$$

Relations (15) and (16) are obtained from (12) and (13) by inversion. For $x = -\frac{1}{2}$, (15) and (16) become (15) and (16), respectively, in [2], which Gould obtained by the same method.

$$2^{n-1}\left\{1 + (-1)^n \binom{x}{n}^{-1}\right\} = \sum_{k=0}^{[n/2]} \frac{\binom{n}{2k}}{\binom{x}{2k}} A_{2k}(x) \frac{x-n}{x-2k} \tag{17}$$

$$(18)\qquad 2^{n-1}\left\{1 - (-1)^n \binom{x}{n}^{-1}\right\} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{\binom{n}{2k+1}}{\binom{x}{2k+1}} A_{2k+1}(x)\, \frac{x - n}{x - 2k - 1} \quad .$$

Relations (17) and (18) are obtained by respectively adding and subtracting (15) and (16). When $x = -\frac{1}{2}$, (17) is equivalent to (17) in [2].

$$(19)\qquad \begin{aligned} (1 - u^2)^{-x-1} \int_0^u t^{-2x-1}(1 - t^2)^x dt &= \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n - 2x)\binom{x}{n}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} A_n(x)}{(x - n)\binom{x}{n}} \cdot \frac{u^{2n-2x}}{(2 - u^2)^{n+1}} \quad . \end{aligned}$$

$$(20)\qquad \int_0^u \frac{t^{-2x-1}}{1 + t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n - 2x)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} A_n(x)}{(x - n)\binom{x}{n}} \cdot \frac{u^{2n-2x}}{(2 + u^2)^{n+1}}$$

$$(21)\qquad A_n(x) = (2n - 2x)\binom{x}{n} \int_0^1 t^{-2x-1}(2t^2 - 1)^n dt \quad .$$

A proof of (19) is indicated below. Let the left-hand side of (19) = y, i.e.,

$$y = \sum_{n=0}^{\infty} \binom{-x-1}{n} (-1)^n u^{2n} \sum_{k=0}^{\infty} \binom{x}{k} \frac{(-1)^k u^{2k-2x}}{(2k - 2x)} = \sum_{n=0}^{\infty} \theta_n u^{2n-2x} \quad ,$$

or

$$\begin{aligned} y &= \sum_{k=0}^{\infty} \binom{x}{k} \frac{(-1)^k u^{2k-2x}}{(2k - 2x)} \sum_{n=k}^{\infty} \binom{-x-1}{n-k} (-1)^{n-k} u^{2n-2k} \\ &= \sum_{n=0}^{\infty} (-1)^n u^{2n-2x} \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{-x-1}{n-k}}{(2k - 2x)} \quad . \end{aligned}$$

We will return to the above expression, but first we will direct our efforts toward finding an expression for θ_n, as defined above. If we differentiate the integral expression for y and its series form:

$$y' = (1 - u^2)^{-x-1} u^{-2x-1}(1 - u^2)^x + 2u(x + 1)(1 - u^2)^{-1} y = \sum_{n=0}^{\infty} (2n - 2x)\,\theta_n u^{2n-2x-1} \quad .$$

$$\therefore \quad (1 - u^2)y' = u^{-2x-1} + 2u(x + 1)y\,;$$

converting this differential equation into series form by means of the foregoing relations, we obtain a recursion: $(n + 1 - x)\theta_{n+1} = (n + 1)\theta_n (n = 0, 1, 2, \cdots)$; $1 = -2x\theta_0$ $(x \neq 0)$. By an easy induction on this last recursion, we obtain the expression

$$\theta_n = (-1)^{n+1} \Big/ 2(x - n)\binom{x}{n}\,;$$

this proves the first identity of (19). We may convert such form as follows, by use of (16):

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n - 2x)\binom{x}{n}} = \sum_{n=0}^{\infty} \frac{2^{-n} u^{2n-2x}}{(2n - 2x)\binom{x}{n}} \binom{x}{n}(x - n) \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{x}{k}} \frac{A_k(x)}{(x - k)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k A_k(x)}{(x - k)\binom{x}{k}} \sum_{n=k}^{\infty} -2^{-n-1}\binom{n}{k} u^{2n-2x}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k A_k(x)}{(x - k)\binom{x}{k}} \sum_{n=0}^{\infty} -2^{-n-k-1}\binom{n + k}{k} u^{2n+2k-2x}$$

$$y = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} A_k(x)\, 2^{-k-1}}{(x - k)\binom{x}{k}} u^{2k-2x} \left(1 - \tfrac{1}{2}u^2\right)^{-k-1} \quad ,$$

which reduces to (19). If we return to the double summation expression for y which we first obtained, we arrive at the interesting identity:

(22) $$\frac{1}{(x - n)\binom{x}{n}} = \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{-x - 1}{n - k}}{x - k} \; .$$

Relation (20) is similarly obtained from (15) being substituted in the first identity of (20), which is readily obtained from the integral expression by direct integration of a geometric series.

Relation (21) is derived from (12), and may be verified by expansion of the integrand in (21), term-by-term integration and comparison with (12).

(23) $$\sum_{n=0}^{\infty} \frac{A_n(x)}{(x - n)\binom{x}{n} n!n!} \left(\tfrac{1}{2}u\right)^{2n} = \sum_{n=0}^{\infty} \frac{J_n(u)\, u^n}{(x - n)n!} \; .$$

Relation (23) is obtained by employing the definition of $J_n(u)$, the ordinary Bessel function of n^{th} order, and relation (12).

$$\int_0^u t^{-2x-1}(1 - t^2)^x dt = \sum_{n=0}^{\infty} (-1)^n \binom{x}{n} \frac{u^{2n-2x}}{2n - 2x} \tag{24}$$

Relation (24) is obtained by expansion of the integrand and term-by-term integration of the result.

By the substitution of $x = -\frac{1}{2}$ in (19)-(21), (23) and (24), we obtain Gould's identities (18), (19), (21) (in variant form), (28) and (35) in [2].

By the substitutions $u = iv$, $t = is$ in (19), (20) and (24) (and reconverting back to the dummy variables u and t), we derive the following:

$$\begin{aligned} (1 + u^2)^{-x-1} \int_0^u t^{-2x-1}(1 + t^2)^x dt &= \sum_{n=0}^{\infty} \frac{u^{2n-2x}}{(2n - 2x)\binom{x}{n}} \\ &= \sum_{n=0}^{\infty} \frac{A_n(x)}{(n - x)\binom{x}{n}} \frac{u^{2n-2x}}{(2 + u^2)^{n+1}} \end{aligned} \tag{25}$$

$$\int_0^u \frac{t^{-2x-1}}{1 - t^2} dt = \sum_{n=0}^{\infty} \frac{u^{2n-2x}}{2n - 2x} = \sum_{n=0}^{\infty} \frac{A_n(x)}{(n - x)\binom{x}{n}} \frac{u^{2n-2x}}{(2 - u^2)^{n+1}} \tag{26}$$

$$\int_0^u t^{-2x-1}(1 + t^2)^x dt = \sum_{n=0}^{\infty} \binom{x}{n} \frac{u^{2n-2x}}{2n - 2x} \tag{27}$$

If, in (19), we make the substitutions

$$\frac{u^2}{2 - u^2} = v^2, \quad \frac{t^2}{2 - t^2} = s^2$$

(and then reconvert to the dummy variables u and t), we obtain:

$$(1 - u^2)^{-x-1} \int_0^u t^{-2x-1} \frac{(1 - t^2)^x}{1 + t^2} dt = \sum_{n=0}^{\infty} (-1)^n \frac{A_n(x)\, u^{2n-2x}}{(2n - 2x)\binom{x}{n}} \tag{28}$$

The "conjugate" of (28) is the following:

$$(29) \qquad (1 + u^2)^{-x-1} \int_0^u t^{-2x-1} \frac{(1 + t^2)^x}{1 - t^2}\, dt = \sum_{n=0}^{\infty} \frac{A_n(x)\, u^{2n-2x}}{(2n - 2x)\binom{x}{n}} .$$

Another genus of relations is obtained by considering variations in form of the basic definition of $g(u,x)$ in (2), or related functions. For example, since

$$(1 + u)^{x+r}(1 + u)^{-x+s} = (1 + u)^{r+s} ,$$

we arrive at

$$(30) \qquad \sum_{k=0}^{n} \binom{x + r}{k}\binom{-x + s}{n - k} = \binom{r + s}{n} .$$

This is simply a special case of the Vandermonde convolution theorem; its chief point of interest here is the invariance of (30) with respect to x. Setting $r = 0$ and $s = -1$, as a special case of (30), we have:

$$(31) \qquad \sum_{k=0}^{n} \binom{x}{k}\binom{-x - 1}{n - k} = \binom{-1}{n} = (-1)^n .$$

By considering the convolution of the expression

$$(1 + u)^{x+r}(1 + u)^{-x+s}(1 - u)^{-1} = (1 + u)^{r+s}(1 - u)^{-1} ,$$

we obtain the following identity:

$$(32) \qquad \sum_{k=0}^{n} \binom{x + r}{n - k} A_k(-r + s) = A_n(r + s) .$$

Again, the interesting point in (32) is the invariance with respect to x of the right member. When $r = 0$, $s = x$, we obtain the expression in (5) for $A_n(x)$. By setting $x = 0$ and $s = x$ in (32), we obtain the recursion:

$$(33) \qquad A_n(x + r) = \sum_{k=0}^{n} \binom{r}{k} A_{n-k}(x) .$$

Another interesting identity displaying invariance on x is obtained by considering the convolution of $(1 + u)^{x+r}(1 - u)^{-1} \cdot (1 + u)^{-x+s}(1 - u)^{-1} = (1 + u)^{r+s}(1 - u)^{-2}$.

$$(34) \qquad \sum_{k=0}^{n} A_k(x+r)A_{n-k}(-x+s) = (n+1)A_n(r+s) - (r+s)A_{n-1}(r+s-1) .$$

As special cases of (34), for $r = 0$, $s = 0$, we obtain:

$$(35) \qquad \sum_{k=0}^{n} A_k(x)A_{n-k}(-x) = n+1 .$$

For $r = 0$, $s = -1$, (34) yields:

$$(36) \qquad \sum_{k=0}^{n} A_k(x)A_{n-k}(-x-1) = 1 + [\tfrac{1}{2}n] .$$

By considering the sum

$$\sum_{k=0}^{r-1} \frac{(1+u)^{x+k}}{1-u} = \frac{(1+u)^x}{1-u} \cdot \frac{(1+u)^r - 1}{u} ,$$

we obtain the following recursion:

$$(37) \qquad \sum_{k=0}^{r-1} A_n(x+k) = \sum_{k=0}^{n} \binom{r}{n-k+1} A_k(x) .$$

For $r = 2$, we obtain as a special case of (37):

$$(38) \qquad A_n(x+1) = A_n(x) + A_{n-1}(x) .$$

We may also derive (38) by letting $r = 1$ in (33). As should otherwise be evident, this is the same recursion satisfied by the binomial coefficients, i.e., if $\binom{x}{n}$ is substituted for $A_n(x)$.

The list of identities in (10)-(38) is by no means exhaustive, and indeed it should have by now become evident to the reader that the variety of derivable identities stemming from the basic definition in (2) is virtually unlimited. As previously intimated, Gould [2] has observed that far more general results are available in the existing literature, and it is primarily for this reason that (10)-(38) have been offered with a minimum of explanation. The real purpose of this paper is to give a proof of (9), and the other identities have been presented solely for the sake of exposition. The proof of (9), which follows, depends on differential equations and the method of equating coefficients. The writer was unable to obtain a more direct proof, and this is left as a project for the interested reader.

We begin by adopting the following definitions, in the interest of simplicity of expression:

$$C_n = A_n(x - \tfrac{1}{2}) ; \quad \overline{C}_n = A_n(-x - \tfrac{1}{2}) \tag{39}$$

$$Q_n = C_n \overline{C}_n \tag{40}$$

$$R_n = Q_n - Q_{n-1} \tag{41}$$

$$J_n = \binom{x - \frac{1}{2}}{n} ; \quad \overline{J}_n = \binom{-x - \frac{1}{2}}{n} \tag{42}$$

$$K_n = J_n \overline{J}_n \tag{43}$$

$$q = K_1 = \tfrac{1}{4} - x^2 \quad . \tag{44}$$

Some useful relations are indicated below, which are evident from the definitions given in (39)-(44):

$$C_n = C_{n-1} + J_n ; \quad \overline{C}_n = \overline{C}_{n-1} + \overline{J}_n \tag{45}$$

$$J_n = \left(\frac{x + \frac{1}{2} - n}{n}\right) J_{n-1} ; \quad \overline{J}_n = \left(\frac{-x + \frac{1}{2} - n}{n}\right) \overline{J}_{n-1} \tag{46}$$

$$K_n = \left\{\frac{(\frac{1}{2} - n)^2 - x^2}{n^2}\right\} K_{n-1} = \left\{\frac{q + n(n - 1)}{n^2}\right\} K_{n-1} \quad . \tag{47}$$

Our aim is to first obtain a recursion for the coefficients Q_n, then to show that the same recursion, with the same initial conditions, is satisfied by the expression in the right member of (9). The following development makes free use of the relations and definitions in (39)-(47):

$$R_n = C_n \overline{C}_n - C_{n-1} \overline{C}_{n-1} = C_n \overline{C}_n - (C_n - J_n)(\overline{C}_n - \overline{J}_n) \quad ,$$

or

$$R_n = J_n \overline{C}_n + \overline{J}_n C_n - K_n \quad . \tag{48}$$

If we increase the subscript in (48) by unity, multiply by $(n + 1)$, and apply (45)-(47), we obtain:

$$(n + 1)R_{n+1} = (n + 1)J_{n+1}(\overline{C}_n + \overline{J}_{n+1}) + (n + 1)\overline{J}_{n+1}(C_n + J_{n+1}) - (n + 1)K_{n+1} \, ,$$

or

$$(n + 1)R_{n+1} = (x - \tfrac{1}{2} - n)J_n \overline{C}_n + (-x - \tfrac{1}{2} - n)\overline{J}_n C_n + (n + 1)K_{n+1} \quad . \tag{49}$$

If we decrease the subscript in (48) by unity, and again use relations (45)-(47), we obtain:

$$R_{n-1} = \overline{C}_{n-1}J_{n-1} + C_{n-1}\overline{J}_{n-1} - K_{n-1} = (\overline{C}_n - \overline{J}_n)\left(\frac{n}{x + \frac{1}{2} - n}\right)J_n$$

$$+ (C_n - J_n)\left(\frac{n}{-x + \frac{1}{2} - n}\right)\overline{J}_n - \left(\frac{n^2}{q + n(n-1)}\right)K_n \quad .$$

Multiplying the above throughout by

$$\frac{(x + \frac{1}{2} - n)(-x + \frac{1}{2} - n)}{n} = \frac{q + n(n-1)}{n}, \quad \frac{(q + n(n-1))}{n}R_{n-1} = (-x + \tfrac{1}{2} - n)(\overline{C}_n - \overline{J}_n)J_n$$

$$+ (x + \tfrac{1}{2} - n)(C_n - J_n)\overline{J}_n - nK_n,$$

or

$$\left(\frac{q}{n} + n - 1\right)R_{n-1} = (-x + \tfrac{1}{2} - n)J_n\overline{C}_n + (x + \tfrac{1}{2} - n)\overline{J}_nC_n + (n-1)K_n \quad . \tag{50}$$

If we now multiply (48) throughout by $2n$ and add this result to the sum of (49) and (50), we obtain the following recursion:

$$(n+1)R_{n+1} + 2nR_n + \left(\frac{q}{n} + n - 1\right)R_{n-1} = (n+1)(K_{n+1} - K_n) \quad . \tag{51}$$

If, in (51), we substitute for R_n the expression $Q_n - Q_{n-1}$ from (41), and similarly for the other subscripts, we obtain a third-order recursion involving the Q_n's:

$$\begin{aligned}(n+1)Q_{n+1} + (n-1)Q_n + \left(\frac{q}{n} - n - 1\right)Q_{n-1} - \left(\frac{q}{n} + n - 1\right)Q_{n-2}\\ = (n+1)(K_{n+1} - K_n) \quad .\end{aligned} \tag{52}$$

This, then, is the recursion which we now seek to demonstrate is also satisfied by the expression in the right member of (9).

We begin by introducing some additional definitions, again for the sake of brevity:

$$P_n = J_n\sum_{k=0}^{n}\overline{J}_{n-k}\frac{x - \frac{1}{2} - n}{x - \frac{1}{2} - k}; \quad \overline{P}_n = \overline{J}_n\sum_{k=0}^{n}J_{n-k}\frac{x + \frac{1}{2} + n}{x + \frac{1}{2} + k} \tag{53}$$

$$h = w(u,x) = -(1+u)^{-x-\frac{1}{2}}\int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt\,; \quad \overline{h} = w(u,-x) \quad . \tag{54}$$

The statement of identity (9) may then be condensed to the simple form:

$$Q_n = \tfrac{1}{2}(P_n + \overline{P}_n) \quad . \tag{55}$$

Since

$$(1+u)^{-x-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-x-\frac{1}{2}}{n}u^n \quad ,$$

and

$$-\int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt = \sum_{n=0}^{\infty} \frac{u^{-x+\frac{1}{2}+n}}{x-\frac{1}{2}-n},$$

$$h = \sum_{n=0}^{\infty} u^{-x+\frac{1}{2}+n} \sum_{k=0}^{n} \binom{-x-\frac{1}{2}}{n-k} \frac{1}{x-\frac{1}{2}-k} = \sum_{n=0}^{\infty} u^{-x+\frac{1}{2}+n} \sum_{k=0}^{n} \frac{\overline{J}_{n-k}}{x-\frac{1}{2}-k}.$$

Comparing the latter expression with the first definition in (53), we have:

$$(56) \qquad h = \sum_{n=0}^{\infty} \frac{P_n u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n}; \quad \text{similarly,} \quad \overline{h} = \sum_{n=0}^{\infty} \frac{\overline{P}_n u^{+x+\frac{1}{2}+n}}{(-x-\frac{1}{2}-n)\overline{J}_n}.$$

By differentiating h, we obtain the expressions:

$$(57) \qquad h' = \sum_{n=-1}^{\infty} \frac{-P_{n+1} u^{-x+\frac{1}{2}+n}}{J_{n+1}} = -P_0 u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{-(n+1)P_{n+1} u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n}$$

$$h'' = (x+\tfrac{1}{2}) P_0 u^{-x-3/2} + \sum_{n=-1}^{\infty} \frac{(n+2)P_{n+2} u^{-x+\frac{1}{2}+n}}{J_{n+1}}$$

$$(58) \qquad = (x+\tfrac{1}{2}) P_0 u^{-x-3/2} + P_1 u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{(n+1)(n+2)P_{n+2} u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n}.$$

On the other hand, if we differentiate h, as defined in (54), we obtain:

$$h' = -(1+u)^{-x-\frac{1}{2}} u^{-x-\frac{1}{2}} (1-u)^{-1} + (x+\tfrac{1}{2})(1+u)^{-x-3/2} \int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt$$

$$= -(1+u)^{-x-\frac{1}{2}} u^{-x-\frac{1}{2}} (1-u)^{-1} - (x+\tfrac{1}{2})(1+u)^{-1} h,$$

or:

$$(59) \qquad (1-u^2)h' + (x+\tfrac{1}{2})(1-u)h + (1+u)^{-x+\frac{1}{2}} u^{-x-\frac{1}{2}} = 0.$$

By a second differentiation,

$$(1-u^2)h'' - 2uh' + (x+\tfrac{1}{2})(1-u)h' - (x+\tfrac{1}{2})h = \left\{\frac{x+\frac{1}{2}}{u} + \frac{x-\frac{1}{2}}{1+u}\right\}(1+u)^{-x+\frac{1}{2}} u^{-x-\frac{1}{2}}$$

$$= -\left\{\frac{x+\frac{1}{2}}{u} + \frac{x-\frac{1}{2}}{1+u}\right\}\left\{(1-u^2)h' + (x+\tfrac{1}{2})(1-u)h\right\},$$

or after simplification:

$$(60) \qquad (u+u^2-u^3-u^4)h'' + \left\{(x+\tfrac{1}{2}) + (3x+\tfrac{1}{2})u - (x+5/2)u^2 - (3x+5/2)u^3\right\}h'$$
$$+ \left\{(x+\tfrac{1}{2})^2 + (x+\tfrac{1}{2})(x-3/2)u - 2(x+\tfrac{1}{2})^2 u^2\right\}h = 0.$$

By means of the series form for h in (56), for h' in (57), and the identity:

$$(1+u)^{-x+\frac{1}{2}}u^{-x-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-x+\frac{1}{2}}{n}u^{-x-\frac{1}{2}+n} = u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty}\frac{(-x+\frac{1}{2})}{n+1}\,\overline{J}_n u^{-x+\frac{1}{2}+n} \quad ,$$

we may convert (59) entirely into series form, with certain manipulations based on properties of the binomial coefficients, designed to maintain the exponent of u in the various expressions the same $(-x + \frac{1}{2} + n,$ in our development), and to contain the factor J_n in the denominator of each expression, which may subsequently be cancelled. If we do so, we obtain the following:

$$\sum_{n=0}^{\infty}\frac{-(n+1)P_{n+1}u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} + \sum_{n=1}^{\infty}\frac{(x+\frac{1}{2}-n)P_{n-1}u^{-x+\frac{1}{2}+n}}{nJ_n}$$

$$+ (x+\tfrac{1}{2})\sum_{n=0}^{\infty}\frac{P_n u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} - (x+\tfrac{1}{2})\sum_{n=1}^{\infty}\frac{P_{n-1}u^{-x+\frac{1}{2}+n}}{nJ_n} + \sum_{n=0}^{\infty}\frac{(-x+\frac{1}{2})}{n+1}\,\overline{J}_n u^{-x+\frac{1}{2}+n}$$

$$= 0 \ .$$

If we now equate the coefficients of similar powers of u in the above expression and simplify by multiplying throughout by $(x - \frac{1}{2} - n)\, J_n$, we obtain:

$$P_1 = q + 2x\ ; \quad -(n+1)P_{n+1} - (x-\tfrac{1}{2}-n)P_{n-1} + (x+\tfrac{1}{2})P_n = (x-\tfrac{1}{2})(x-\tfrac{1}{2}-n)\frac{K_n}{n+1},$$

or

(61) $$(n+1)(P_{n+1} - P_{n-1}) - (x+\tfrac{1}{2})(P_n - P_{n-1}) = \left\{\frac{q}{n+1} + x - \tfrac{1}{2}\right\}K_n \qquad (n = 1, 2, \cdots)$$

$$P_0 = 1, \qquad P_1 = q + 2x \quad .$$

By a similar, though more complicated manipulation of the series in (56)-(58), we may express (60) in series form, yielding another recursion for the P_n's. The development is omitted here, since it is somewhat lengthy. The interested reader may, with a little elbow grease, verify that the following form of the recursion is first obtained:

$$\frac{-(n+1)^2}{x-\frac{1}{2}-n}P_{n+1} + \left\{\frac{x(n+2)+q+n^2+\frac{1}{2}n}{x-\frac{1}{2}-n}\right\}P_n + \left\{\frac{xn-q-n^2-\frac{1}{2}n}{n}\right\}P_{n-1}$$

$$+ \frac{q+n^2-n}{n}P_{n-2} = 0\ , \qquad (n = 2, 3, 4, \cdots)\ ,$$

with

$$P_1 = q + 2x, \qquad P_0 = 1, \qquad P_2 = (\tfrac{1}{2}q - 1)^2 + xq \ .$$

By multiplying the latter expression throughout by $n(x - \frac{1}{2} - n)$, simplifying the result, and shifting the terms around, as the reader may verify, the following form of the recursion is obtained:

$$\begin{aligned}(62)\qquad & -n(n+1)^2(P_{n+1} - P_{n-1}) + n(q + n(n+\tfrac{1}{2}))(P_n - P_{n-2}) \\ & + (\tfrac{1}{2}q - n(n+\tfrac{1}{2}))(P_{n-1} - P_{n-2}) + \{n(n+2)(P_n - P_{n-1}) - (q + n(n-1))(P_{n-1} - P_{n-2})\}x \\ & = 0\end{aligned}$$

(for $n = 2, 3, \cdots$); with $P_0 = 1$, $P_1 = q + 2x$, $P_2 = (\frac{1}{2}q - 1)^2 + xq$.

We may further simplify (61) and (62), if we introduce another symbol:

$$(63)\qquad V_n = P_n - P_{n-1},$$

which also yields:

$$V_n + V_{n+1} = P_{n+1} - P_{n-1}.$$

By substituting the appropriate expressions in (61) and (62), we obtain:

$$(64)\qquad (n+1)(V_{n+1} - K_{n+1}) = (x - \tfrac{1}{2} - n)(V_n + K_n)$$

(making use of the relation

$$\frac{q}{n+1}K_n = (n+1)K_{n+1} - nK_n,$$

a variant of (47)), and

$$\begin{aligned}(65)\qquad & -n(n+1)^2(V_n + V_{n+1}) + n(q + n(n+\tfrac{1}{2}))(V_{n-1} + V_n) + (\tfrac{1}{2}q - n(n+\tfrac{1}{2}))V_{n-1} \\ & + n(n+2)\,x\,V_n - (q + n(n-1))\,x\,V_{n-1} = 0.\end{aligned}$$

Rearranging the terms in (64), we obtain an expression for xV_n:

$$(66)\qquad xV_n = -xK_n + (n+\tfrac{1}{2})(V_n + K_n) + (n+1)(V_{n+1} - K_{n+1}).$$

If we substitute the expression in (66) and the corresponding expression with the subscript reduced by unity in (65), again use (47) in variant form, and simplify, (65) is transformed to the following form:

$$(67)\qquad (n+1)V_{n+1} + 2nV_n + \left(\frac{q}{n} + n - 1\right)V_{n-1} = (n+1)(K_{n+1} - K_n) + 2xK_n.$$

The reader may verify the simplification to the above form, using the indicated procedure. If we now replace the V_n's in (67) by the corresponding P_n's, in accordance with (63), we readily obtain the following recursion involving the P_n's:

$$\begin{aligned}(68)\qquad & (n+1)P_{n+1} + (n-1)P_n + \left(\frac{q}{n} - n - 1\right)P_{n-1} - \left(\frac{q}{n} + n - 1\right)P_{n-2} \\ & = (n+1)(K_{n+1} - K_n) + 2xK_n.\end{aligned}$$

If we replace x by -x in (68), observing that q and the K_n's are even functions of x, we obtain the "conjugate" of (68):

$$(69)\qquad \begin{aligned}(n+1)\overline{P}_{n+1} + (n-1)\overline{P}_n + \left(\frac{q}{n} - n - 1\right)\overline{P}_{n-1} - \left(\frac{q}{n} + n - 1\right)\overline{P}_{n-2} \\ = (n+1)(K_{n+1} - K_n) - 2xK_n \ .\end{aligned}$$

If we add (68) and (69) and divide by 2, we obtain the following recursion involving $\frac{1}{2}(P_n + \overline{P}_n)$, the terms involving x cancelling:

$$(70)\qquad \begin{aligned}(n+1)\tfrac{1}{2}(P_{n+1} + \overline{P}_{n-1}) + (n-1)\tfrac{1}{2}(P_n + \overline{P}_n) + \left(\frac{q}{n} - n - 1\right)\tfrac{1}{2}(P_{n-1} + \overline{P}_{n-1}) \\ - \left(\frac{q}{n} + n - 1\right)\tfrac{1}{2}(P_{n-2} + \overline{P}_{n-2}) = (n+1)(K_{n+1} - K_n) \ .\end{aligned}$$

Comparing (52) with (70), we see that $\frac{1}{2}(P_n + \overline{P}_n)$ satisfies the same recursion as Q_n. We need to demonstrate only that $Q_n = \frac{1}{2}(P_n + \overline{P}_n)$ for n = 0, 1, and 2, to complete the proof of (55), i.e., (9). We have already demonstrated that

$$P_0 = 1, \qquad P_1 = q + 2x, \qquad P_2 = (\tfrac{1}{2}q - 1)^2 + qx \ .$$

Therefore,

$$\begin{aligned}\tfrac{1}{2}(P_0 + \overline{P}_0) = 1 \ ; \quad \tfrac{1}{2}(P_1 + \overline{P}_1) = \tfrac{1}{2}(q + 2x + q - 2x) = q \ ; \quad \tfrac{1}{2}(P_2 + \overline{P}_2) \\ = \tfrac{1}{2}\left\{(\tfrac{1}{2}q - 1)^2 + qx + (\tfrac{1}{2}q - 1)^2 - qx\right\} = (\tfrac{1}{2}q - 1)^2 \ .\end{aligned}$$

We may verify that

$$C_0 = 1, \qquad C_1 = \tfrac{1}{2} + x, \qquad C_2 = \tfrac{1}{2}x^2 + 7/8 = 1 - \tfrac{1}{2}q \ ,$$

from (5), substituting $x - \frac{1}{2}$ for x. Then

$$Q_0 = 1, \qquad Q_1 = (\tfrac{1}{2} + x)(\tfrac{1}{2} - x) = q, \qquad Q_2 = (1 - \tfrac{1}{2}q)^2 \ .$$

This completes the proof of (9).

The limits of convergence of the power series in this paper have been ignored, since we have treated these series as formal generating functions of the coefficients under investigation.

It was initially remarked that this study was originally motivated by a desire to find an expression for $A_n^2(x)$ in single-summation form, and that this effort was unsuccessful. However, certain results were obtained which suggest areas of investigation for the interested reader. A recursion for the $A_n^2(x)$'s may be derived in the following manner. We begin by introducing a new definition:

$$(71)\qquad T_n = A_n^2(x) - A_{n-1}^2(x) \ .$$

By using the property

$$A_n(x) - A_{n-1}(x) = \binom{x}{n},$$

and recursion (38), we may obtain an alternate expression for T_n:

$$T_n = \{A_n(x) - A_{n-1}(x)\}\{A_n(x) + A_{n-1}(x)\} ,$$

or

(72) $$T_n = \binom{x}{n} A_n(x + 1) .$$

Therefore,

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \left(A_k^2(x) - A_{k-1}^2(x) \right) = \sum_{k=1}^{n} \binom{x}{k} A_k(x + 1) ,$$

which yields:

(73) $$A_n^2(x) = \sum_{k=0}^{n} \binom{x}{k} A_k(x + 1) .$$

Of course, there is the more obvious identity:

(74) $$A_n^2(x) = A_n(x) \sum_{k=0}^{n} \binom{x}{k} ,$$

which is simply (5) multiplied by $A_n(x)$.

Neither (73) nor (74), however, are single-summation expressions, since they involve the coefficient $A_k(x)$ (or $A_n(x)$), which is itself a single-summation expression.

The recursion for the $A_n^2(x)$'s is obtained by substituting

$$\frac{T_n}{\binom{x}{n}}$$

for $A_n(x + 1)$ in:

$$(n + 1)A_{n+1}(x + 1) - (x + 2)A_n(x + 1) + (x + 1 - n)A_{n-1}(x + 1) = 0 ,$$

which is simply (10) with $x + 1$ replacing x. By eliminating the combinatorial terms, we first obtain a second-order recursion involving the T_n's:

(75) $$n(n + 1)^2 T_{n+1} - n(x + 2)(x - n)T_n + (x - n)(x + 1 - n)T_{n-1} = 0 .$$

By substituting the expression in (71) for the T_n's in (75), we are led to the required recursion:

(76) $$\begin{aligned} n(n + 1)^2 A_{n+1}^2(x) + \{n(x - n) - (x + 1)^2\}\{nA_n^2(x) - (x - n)A_{n-1}^2(x)\} \\ - (x - n)(x + 1 - n)^2 A_{n-2}^2(x) = 0 \end{aligned}$$

It should be observed that if the substitution $x = -\frac{1}{2}$ is made in (76), and the substitution $x = 0$ is made in (52), the same recursion results in either case, namely:

$$n(n+1)^2A_{n+1}^2 - (n^2 + \tfrac{1}{2}n + \tfrac{1}{4})(nA_n^2 + (n + \tfrac{1}{2})A_{n-1}^2) + (n + \tfrac{1}{2})(n - \tfrac{1}{2})^2A_{n-2}^2 = 0 . \tag{77}$$

(It is not immediately obvious that (52) reduces to (77) for $x = 0$, but if we observe that, in such case, $Q_n = A_n^2$ and $q = \frac{1}{4}$, we may use known relations to show that

$$(n + 1)(K_{n+1} - K_n)$$

may be expressed in the form:

$$\frac{1}{4n + 2}\left\{(8n^2 + 14n + 6)Q_{n+1} - (4n + 3)Q_n - (8n^2 + 10n + 3)Q_{n-1}\right\} .$$

Substituting this expression in (52), we obtain a form free of terms involving K_n's which reduces to (77).)

In passing, we leave the reader with one possible form of expression for $A_n^2(x)$, which is suggested below by indicating the first few terms:

$$A_n^2(x) = \binom{2n}{n}\left\{\binom{x}{2n} + \tfrac{1}{2}(n + 2)\binom{x}{2n - 1} + \left(\frac{n^3 + 2n^2 + 3n - 4}{8n - 4}\right)\binom{x}{2n - 2} + \cdots\right\} . \tag{78}$$

It is not difficult to prove (78) by induction, as far as it goes, but the subsequent terms become increasingly obscure, as the difficulty in obtaining them also increases. The writer failed to see any pattern in the terms of (78), but that is not to say that one does not exist.

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