SOME SIMPLE SIEVES

R. G. BUSCHMAN
University of Wyoming, Laramie, Wyoming

1. INTRODUCTION

An ancient process for generating the sequence of prime numbers is known by the name of The Sieve of Eratosthenes. This method is presented in many textbooks and is rather widely known. However, it seems to be less widely known that, with some modifications, other interesting sequences can be generated by essentially a sieve process. In particular, we can obtain the sequence of values for some of the common arithmetic functions. First we shall discuss this Sieve of Eratosthenes and some of its modifications, then we will proceed to some "sieves" for generating other sequences.

2. THE SIEVE OF ERATOSTHENES AND MODIFICATIONS

We recall that in order to obtain the sequence of primes by this method, the sequence of integers greater than 1 is first written down. Starting with 2 we then put a slash through each second number beyond 2 in this sequence. This leaves 3 as the first number beyond 2 which is not crossed out, so that we then put a slash through every third number beyond 3. (Note that, for example, 6 now has been crossed out twice.) Since 5 is the first number beyond 3 which is not yet crossed out, we next put a slash through each fifth number beyond 5, and continue in a similar manner. Those numbers remaining (not crossed out) are primes.

In order to place this process in a setting which is more suitable for generalization, we will now modify the process in order to generate the sequence of values of what we shall call the characteristic function for prime numbers, denoted by χ_p. This function has the values $\chi_p(n) = 1$ if n is a prime; $\chi_p(n) = 0$ otherwise. In Table 1 the construction of the successive sequences is illustrated. This table is headed by the sequence of natural numbers in natural order which will thus indicate the position numbers for the elements of the sequences.

Table 1
Successive Sequences for $\chi_p(n)$, $N = 33$

										1										2										3			
	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
$A^{(0)}$	0	$\underline{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$A^{(1)}$	0	1	$\underline{1}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$A^{(2)}$	0	1	1	0	$\underline{1}$	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
$A^{(3)}$	0	1	1	0	1	0	$\underline{1}$	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

The entries in Table 1 are prepared as follows. For the initial sequence, $A^{(0)}$, we enter 0 in the first position and 1 otherwise for this first row of the table and for $n \le N$. The case $N = 33$ is illustrated. In order to begin the process of sieving, we locate the position of the first non-zero element and denote this position by a_1 $(= 2)$ and then convert the entries in positions ma_1, $m = 2, 3, 4, \cdots$ (every second entry beyond 2) from 1 to 0. The resulting sequence is denoted by $A^{(1)}$. The position number of the first non-zero entry beyond position a_1 in $A^{(1)}$ is denoted by a_2 $(= 3)$ and every entry in position ma_2, $m = 2, 3, 4, \cdots$ (every third entry beyond 3) is converted from 1 to 0, if the entry is not already 0, in order to produce the sequence $A^{(2)}$. In general, in the sequence $A^{(k-1)}$ we locate the position of the first non-zero entry beyond position a_{k-1} and denote its position by a_k. Then every entry in position ma_k, $m = 2, 3, 4, \cdots$ (every a_k-th entry beyond a_k) is converted from 1 to 0, if it is not already 0. This produces the sequence $A^{(k)}$.

It is worth noting that, of course, the process can be terminated at $A^{(k-1)}$ if $a_k > \sqrt{N}$ and the sequence $A^{(k-1)}$ coincides with the sequence χ_p for $n \le N$; that is, $a_k = p_k$. The reason that this termination is possible is that the smallest number which a_k actually sieves out is a_k^2, since $a_k a_j$ for $a_j < a_k$ has been sieved out at an earlier step.

In this construction the actual sieving out of the number n is indicated by the conversion of an entry 1 to an entry 0 in position n of the sequence. If a 0 has already appeared, this indicates that the number had been sieved out at a previous step; that is, the number actually possessed a smaller prime factor than the number currently being used as the sieving number. The entire process involves (1) the location of a non-zero element, (2) a counting process, and (3) a change of entry. We note that no divisibility checks are used. One further comment. This process does not involve using any of the sequences $A^{(m)}$ for $m < k - 1$, but only the sequence $A^{(k-1)}$ to produce $A^{(k)}$. As a result, those preceding sequences need not be saved. Even though the original process is ancient, in this form it is quite adaptable for digital computers.

For some purposes a simpler sieve which yields slightly different information is of value; this is illustrated in Table 2.

Table 2

First Modification, $N = 33$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$A^{(0)}$	1	$\underline{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$A^{(1)}$	1	0	$\underline{1}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$A^{(2)}$	1	0	0	0	$\underline{1}$	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
$A^{(3)}$	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

The change of the results is indicated by the appearance of 1 in position 1 and in the appearance of 0 in position p_k for primes $p_k \le \sqrt{N}$. Hence the result is the sequence of values of the characteristic function for the set which contains 1 and those primes satisfying the inequality $\sqrt{N} < p_k \le N$. The process is simpler in the sense that for the general step we sieve out the numbers ma_k for $m = 1, 2, 3, 4, \cdots$; that is, each a_k-th number, counting from the beginning. Here the process <u>must</u> be stopped at the sequence $A^{(k-1)}$ if $a_k > \sqrt{N}$.

These first two processes have the disadvantage in that multiple sieving of elements occurs whenever the position number is composite. The following method will eliminate this problem for the modified sieve, although it is a much more complicated procedure.

Consider the sequence $A^{(0)}$ where $a_k = 1$ for all k. Since the first 1 beyond position 1 which appears is in position a_1 (= 2) we sieve each second element by subtracting 1 from the entries in position ma_k, $m = 1, 2, 3, 4, \cdots$, then we rename the resulting sequence $A^{(1)}$. Next, the first 1 which appears beyond position 1 in $A^{(1)}$ is in position a_2 (= 3) and we use the sequence $A^{(1)}$ itself to generate the sequence of elements which are to be sieved out of $A^{(1)}$ to produce $A^{(2)}$. Subtract the entry in position m of $A^{(1)}$ from the entry in position ma_2 for $m = 1, 2, 3, 4, \cdots$ provided $ma_2 \le N$. We note that if 2 divides m, then $m = 2m'$ and in position $ma_2 = 2m'a_2$ the entry is 0 so that we subtract 0 from 0. (This replaces the operation of leaving 0 as 0.) In general, we locate the first non-zero entry beyond position 1 in $A^{(k-1)}$ and denote this position by a_k, then we subtract the entry in position m of $A^{(k-1)}$ from the entry in position ma_k for $m = 1, 2, 3, 4, \cdots$ with $ma_k \le N$. The process is stopped at the sequence $A^{(k-1)}$ if $a_k > \sqrt{N}$ and the final result of this second modification is the same as that of the first modification. The details are illustrated successively in Table 3.

Table 3
Second Modification, $N = 33$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$A^{(0)}$	1	$\underline{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
		$-\bar{1}$		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1	
$A^{(1)}$	1	0	$\underline{1}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
			$-\bar{1}$			-0			-1			-0			-1			-0			-1			-0			-1			-0			-1
$A^{(2)}$	1	0	0	0	$\underline{1}$	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
					$-\bar{1}$					-0					-0					-0					-1					-0			
$A^{(3)}$	1	0	0	0	0	0	$\underline{1}$	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

A useful way of thinking of the process is that $A^{(k-1)}$ is expanded by the factor a_k and subtracted from itself.

Another method closely related to this second modification and which eliminates multiple sieving is discussed by G. S. Arzumanjan [1]. A geometric construction for the second modification is given in [3].

Although these two modifications are not perhaps very important for the problem of generating primes, they have been presented in detail because of their similarity to sieves for other sequences which will be discussed in the next section.

3. SEQUENCES OF VALUES OF CERTAIN ARITHMETIC FUNCTIONS

The sequences which are to be discussed in this section are computed without advance knowledge of a table of primes; that is, the generation of the sequence of primes is contained internally in the process, as needed.

The first function which we will consider is the number of distinct prime factors of n which we denote by $\omega(n)$. We generate the sequence of values of the function ω by a slight alteration of that first modification for the generation of χ_p which was discussed in Sec. 1. In the case of primes we can think of the composites as falling through the sieve and being discarded; this present alteration can be thought of as collecting those things which fall through our sieve in little boxes. In order to indicate how this process goes, let $B^{(0)}$ be the sequence with 0 in each position. To produce $B^{(1)}$ we add 1 to each second entry of $B^{(0)}$. The first 0 which appears beyond position 1 of $B^{(1)}$ is in position p_2 ($= 3$). We next add 1 to each entry in $B^{(1)}$ in position mp_2, $m = 1, 2, 3, 4, \cdots$ to produce $B^{(2)}$. Continuing in this manner we can state the general step of this iterative process. Locate the first 0 which appears beyond position 1 in $B^{(k-1)}$, this will be in position p_k. Next add 1 to each entry in $B^{(k-1)}$ in position mp_k, $m = 1, 2, 3, 4, \cdots$ for $mp_k < N$. This will produce the sequence $B^{(k)}$. The process can be stopped at sequence $B^{(k-1)}$ if $2p_k > N$, since thereafter only one entry, the entry in position p_k, will be altered. The remaining entries which are 0 and are beyond position 1 now indicate primes satisfying $N/2 < p \le N$, if $N \ge 4$, and hence the process can be completed by converting the 0 entries to 1 in these positions $N/2 < n \le N$ of $B^{(k-1)}$. The sequence $B^{(k)}$ coincides with the sequence for ω for $n \le N$. The results for $N = 33$ are given in Table 4.

Table 4
Sequences for $\omega(n)$, $N = 33$

										1										2										3			
	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
$B^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		$\bar{1}$		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
$B^{(1)}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
			$\bar{1}$			1			1			1			1			1			1			1			1			1			1
$B^{(2)}$	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1
					$\bar{1}$					1					1					1					1					1			
$B^{(3)}$	0	1	1	1	1	2	0	1	1	2	0	2	0	1	2	1	0	2	0	2	1	1	0	2	1	1	1	1	0	3	0	1	1
							$\bar{1}$							1							1							1					
$B^{(4)}$	0	1	1	1	1	2	1	1	1	2	0	2	0	2	2	1	0	2	0	2	2	1	0	2	1	1	1	2	0	3	0	1	1
											$\bar{1}$											1											1
$B^{(5)}$	0	1	1	1	1	2	1	1	1	2	1	2	0	2	2	1	0	2	0	2	2	2	0	2	1	1	1	2	0	3	0	1	2
													$\bar{1}$													1							
$B^{(6)}$	0	1	1	1	1	2	1	1	1	2	1	2	1	2	2	1	0	2	0	2	2	2	0	2	1	2	1	2	0	3	0	1	2
																	$\bar{1}$ $\longrightarrow$																Stop
ω	0	1	1	1	1	2	1	1	1	2	1	2	1	2	2	1	1	2	1	2	2	2	1	2	1	2	1	2	1	3	1	1	2

Since we are adding 1 to the entry at position $n = mp_k$ at the k^{th} step, we count the factor p_k of n exactly once. Hence we have generated the sequence of values for the function ω; that is, $\omega(n)$ appears in position n.

It now becomes an analogous exercise to obtain a sieve process for computing the sequence of values $\tau(n)$, the number of divisors of n. Similarly, the sequence of values $\sigma(n)$, the sum of the divisors of n, can then be computed.

If we next attempt to compute the values $\Omega(n)$, the total number of prime factors of n, the procedure seems to become more complicated. However, one way to proceed is as follows, starting with the sequence $C^{(0)}$ for which all entries are 0. In order to construct $C^{(1)}$ we want to first add 1 to every second entry to count the factor 2^1, then further add 1 to every fourth entry to count the factor 2^2, etc., until $2^k > N$. This subprocess for counting the factors 2 reminds one of the second modification in Sec. 1, if we consider that the first subsequence $C_1^{(1)}$ to be added to $C^{(0)}$ is constructed by entering in position 2m, $m = 1, 2, 3, 4, \cdots$, the value 1, then the second subsequence, $C_2^{(1)}$ which is to be added to $C^{(0)}$ is constructed by entering in position 2m the value from position m of $C_1^{(1)}$, etc. The value 0 is entered otherwise in the subsequences. $C^{(1)}$ is then constructed by adding successively $C_1^{(1)}, C_2^{(1)}, \cdots$ to $C^{(0)}$. From this discussion we can see that if 2^a divides n, then an addition of 1 is carried out in position n for $k = 1, 2, \cdots, a$; thus the process counts the number of prime factors 2 of n. The steps are illustrated in Table 5.

Table 5
Sequences for $\Omega(n)$, N = 33

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$C^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		$\overline{1}$		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
				$\overline{1}$				1				1				1				1				1				1				1	
								$\overline{1}$								1								1								1	
																$\overline{1}$																1	
																																$\overline{1}$	
$C^{(1)}$	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	5	0
			$\overline{1}$			1			1			1			1			1			1			1			1			1			1
									$\overline{1}$									1									1						
																											$\overline{1}$						
$C^{(2)}$	0	1	1	2	0	2	0	3	2	1	0	3	0	1	1	4	0	3	0	2	1	1	0	4	0	1	3	2	0	2	0	5	1
					$\overline{1}$					1					1					1					1					1			
																									$\overline{1}$								
$C^{(3)}$	0	1	1	2	1	2	0	3	2	2	0	3	0	1	2	4	0	3	0	3	1	1	0	4	2	1	3	2	0	3	0	5	1
							$\overline{1}$							1							1							1					
$C^{(4)}$	0	1	1	2	1	2	1	3	2	2	0	3	0	2	2	4	0	3	0	3	2	1	0	4	2	1	3	3	0	3	0	5	1
											$\overline{1}$											1											1
$C^{(5)}$	0	1	1	2	1	2	1	3	2	2	1	3	0	2	2	4	0	3	0	3	2	2	0	4	2	1	3	3	0	3	0	5	2
													$\overline{1}$													1							
$C^{(6)}$	0	1	1	2	1	2	1	3	2	2	1	3	1	2	2	4	0	3	0	3	2	2	0	4	2	2	3	3	0	3	0	5	2
																	$\longrightarrow$																Stop
Ω	0	1	1	2	1	2	1	3	2	2	1	3	1	2	2	4	1	3	1	3	2	2	1	4	2	2	3	3	1	3	1	5	2

To form $C^{(2)}$ from $C^{(1)}$ we locate the first 0 beyond position 1, which is at position p_2 (= 3), and repeat the subprocess, but now using p_2; that is, $C_1^{(2)}$ is the sequence with 1 in position mp_2, $m = 1, 2, 3, 4, \cdots$; $C_2^{(2)}$ is the sequence with 1 in position mp_2^2, $m = 1, 2, 3, 4, \cdots$; etc. Then $C^{(2)}$ is formed by the addition of $C_1^{(2)}, C_2^{(2)}, \cdots$ to $C^{(1)}$. $C_2^{(2)}$ could also be constructed from $C_1^{(2)}$ by entering in position mp_2 the entry from position m of $C_1^{(2)}$ and 0 otherwise.

The general step is somewhat complicated in its description. Consider $C^{(k-1)}$ and locate the first 0 entry beyond position 1; this will be in position p_k by analogy to the sieve for primes. To begin the subprocess, we form $C_1^{(k)}$ by entering 1 in position mp_k, $m = 1, 2, 3, 4, \cdots$, until $mp_k > N$. The subsequence $C_j^{(k)}$ is formed from the subsequence $C_{j-1}^{(k)}$ by entering in position mp_k, $m = 1, 2, 3, \cdots$, the entry from position m of $C_{j-1}^{(k)}$ and 0 otherwise until $mp_k^j > N$. The sequences $C_j^{(k)}$ are successively added to $C^{(k-1)}$ to produce $C^{(k)}$. (It is merely for display purposes that the subsequences are formed separately and then added, the addition process, of course, can be carried out as one progresses.) The process can be stopped at the same point in the computation as for the values of ω and the remaining 0 entries converted to 1, using the same reasoning. It is not difficult to see that we actually have obtained the values $\Omega(n)$.

A slight modification of the entries leads to the sequence $\lambda(n) = (-1)^{\Omega(n)}$, another function of interest in the theory of numbers. After the methods outlined above, this becomes an exercise.

4. THE NUMBER OF REPRESENTATIONS OF A NUMBER AS A PRODUCT OF NUMBERS CONTAINED IN A GIVEN SET

We shall next consider the following problem. Given a subsequence S of natural numbers $1 < a_1 < a_2 < \cdots$, either finite or infinite, we wish to compute the number of distinct representations of a number n as the product of elements of the set S; that is, we wish to compute the number of distinct (except for order) representations of n in the form

$$n = a_1^{b_1} a_1^{b_2} \cdots a_k^{b_k} ,$$

where the a's belong to S and the b's are positive integers. We assume that the set S has been generated separately and we let R(n) denote the number of such representations of n. The sequence of values of R is to be generated by a modified sieve method.

The actual process is somewhat analogous to the procedure of Sec. 3 for computing the sequence Ω. We let $R^{(0)}$ denote the sequence with 1 in position 1 and 0 otherwise. In order to obtain the sequence $R^{(1)}$ we bring down 1 into position 1 and then add the entry from position 1 of $R^{(1)}$ to the entry at position $1a_1$ of $R^{(0)}$ and enter the sum in position a_1 of $R^{(1)}$, then the entry from position 2 of $R^{(1)}$ is added to the entry of position $2a_2$ of $R^{(0)}$ and the sum is entered in position $2a_2$ of $R^{(1)}$, etc., but otherwise the entry at position k of $R^{(1)}$ is taken as the entry at position k of $R^{(0)}$. This set of subprocesses is stopped when $ma_1 > N$. To continue, the entry from position 1 of $R^{(1)}$ is entered in position 1 of $R^{(2)}$ and the entry from position 1 of $R^{(1)}$ is added to the entry from position $1a_2$ of $R^{(1)}$ and the sum is entered in position a_2 of $R^{(2)}$ and successively in order for $m = 1, 2, 3, \cdots$, the entry at position m of $R^{(2)}$ is added to the entry at position ma_2 of $R^{(1)}$ and the sum entered in position ma_2 of $R^{(2)}$ to produce the sequence $R^{(2)}$. (The other entries are carried from $R^{(1)}$ to $R^{(2)}$, addition only takes place at the positions ma_2.) For the general iterative step, in order to obtain $R^{(k)}$ from $R^{(k-1)}$, we add the entry from position m of $R^{(k)}$ to the entry from position ma_k of $R^{(k-1)}$ and enter the sum in position ma_k of $R^{(k)}$, running successively through $m = 1, 2, 3, \cdots$ and stopping if $ma_k > N$. The process terminates at $R^{(k-1)}$ if $a_k > N$. In Table 6 we have chosen $S = \{2, 3, 4, 5, 12, 30, 72\}$ and we indicate the steps of the computation. Note the iterative process which occurs within the computation for each sequence.

Table 6
$R(n)$ for $S = \{2, 3, 4, 5, 12, 30, 72\}$, $N = 33$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$R^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R^{(1)}$	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$R^{(2)}$	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0
$R^{(3)}$	1	1	1	2	0	1	0	2	1	0	0	2	0	0	0	3	0	1	0	0	0	0	0	2	0	0	1	0	0	0	0	3	0
$R^{(4)}$	1	1	1	2	1	1	0	2	1	1	0	2	0	0	1	3	0	1	0	2	0	0	0	2	1	0	1	0	0	1	0	3	0
$R^{(5)}$	1	1	1	2	1	1	0	2	1	1	0	3	0	0	1	3	0	1	0	2	0	0	0	3	1	0	1	0	0	1	0	3	0
R	1	1	1	2	1	1	0	2	1	1	0	3	0	0	1	3	0	1	0	2	0	0	0	3	1	0	1	0	0	2	0	3	0

The iterative procedure from $R^{(k-1)}$ to $R^{(k)}$ can be expressed in terms of the following equations which are to be applied successively for $n = 1, 2, 3, \cdots$ in that order.

$$R^{(k)}(n) = R^{(k-1)}(n), \qquad \text{if } n \neq ma_k ,$$
$$R^{(k)}(n) = R^{(k-1)}(n) + R^{(k)}(m), \qquad \text{if } n = ma_k .$$

For example we have, since $a_4 = 5$,

$$R^{(4)}(20) = R^{(3)}(20) + R^{(4)}(4) = 0 + 2 = 2 ,$$
$$R^{(4)}(21) = R^{(3)}(21) = 0 .$$

Special cases of interest are obtained if $a_k = p_k$, then, of course, $R(n) = 1$ since the representation is unique; if $a_k = k$, then $R(n)$ denotes the number of factorizations of n into integers; if $a_k = k^2$, then $R(n)$ denotes the number of factorizations of n into perfect squares; and if $a_k = p_k^2$, then $R(n)$ is the characteristic function for squares.

5. SOME FURTHER DIRECTIONS

The sequence of lucky numbers has been generated by a sieve technique and some of the properties of this sequence have been ovtained [4, 6]. The question concerning the number of distinct representations of n as a produce of lucky numbers can be approached by means of Sec. 4. A mixed technique of alternately sieving and summing which will generate the sequence of k^{th} powers is due to Moessner [8]; this is discussed and further references are given in a recent paper by C. T. Long [7]. Beginning with V. Brun [2] techniques involving double sieving and other modifications have been used to study the twin prime problem, the Goldbach conjecture, and other problems. An interesting article by David Hawkins [5] on the sieve of Eratosthenes, random sieves, and other matters is well worth consulting.

REFERENCES

1. G. S. Arzumanjan, "On a Modification of the Sieve Method in the Theory of Prime Numbers," {Russian}, Eravan Armjan. Gos. Ped. Inst. im Abovjana. Učen. Zap. Ser. Mat. Nauk (1963), part 2, pp. 31-65.
2. V. Brun, "Le Crible d'Eratosthenes et le Théorème de Goldbach," Norske Vid. Selsk. Skr. Mat.-naturv. Kl. (1920), No. 3, 36 pp.
3. R. G. Buschman, "Pencils of Rays and the Sieve of Eratosthenes," submitted for pub.
4. V. Gardiner, R. Lazarus, M. Metropolis, and S. Ulam, "On Certain Sequences of Integers Defined by Sieves," Math. Mag. 29 (1955/56), pp. 117-122.
5. David Hawkins, "Mathematical Sieves," Sci. Amer. 199 (Dec. 1958), pp. 105-112.
6. D. Hawkins and W. E. Briggs, "The Lucky Number Theorem," Math. Mag. 31 (1957/58), pp. 277-280.
7. C. T. Long, "On the Moessner Theorem on Integral Powers," Amer. Math. Monthly, 73 (1966), pp. 846-851.
8. A. Moessner, Eine Bemerkung über die Potenzen der Naturalichen Zahlen, Bayer. Akad. Wiss. Math-Nat. Kl. S.-B. (1951), p. 29.

FIBONACCI NOTE SERVICE

The Fibonacci Quarterly is offering a service in which it will be possible for its readers to secure background notes for articles. This will apply to the following:

(1) Short abstracts of extensive results, derivations, and numerical data.

(2) Brief articles summarizing a large amount of research.

(3) Articles of standard size for which additional background material may be obtained.

Articles in the Quarterly for which this note service is available will indicate the fact, together with the number of pages in question. Requests for these notes should be made to

Brother Alfred Brousseau
St. Mary's College
Moraga, California 94574

The notes will be Xeroxed.

The price for this service is four cents a page (including postage, materials and labor.)