REPRESENTATIONS AS PRODUCTS OR AS SUMS

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1. INTRODUCTION

In a previous paper [1] the idea of a "sieve" was extended in order to give a method for the computation of the sequences of values for certain functions which occur in the theory of numbers. Some of the important functions are generated as the sequences of coefficients of suitable Dirichlet series or of suitable power series; see, for example, G. H. Hardy and E. M. Wright [2, Chapters 17 and 19]. We will consider the similar problems of the number of representations of an integer as a product with factors chosen from a given set of positive integers and the problem of the number of representations of an integer as a sum with terms chosen from a given set of positive integers. Although quite analogous, the two problems are rarely mentioned together.

To be specific, we consider a subsequence $S = \{a_n\}$, of positive integers, finite or infinite, which satisfies the conditions $a_1 < a_2 < a_3 < \cdots$ with $1 < a_1$ for the case of products and $0 < a_1$ for the case of sums. Our problems can then be stated as (1) compute the number, $R(n)$, of distinct representations of a number n as a product of the form

$$n = a_j^{b_j} a_k^{b_k} \cdots ,$$

with $a_i \in S$, $b_i > 0$, and (2) compute the number, $P(n)$, of distinct representations of a number n as a sum of the form $n = b_j a_j + b_k a_k + \cdots$, $a_i \in S$, $b_i > 0$. Analogous problems are obtained by the restriction $b_i = 1$; these are (1') compute the number, $R'(n)$, of distinct representations of a number n as a product of the form $n = a_j a_k, \cdots, a_i \in S$, that is, of distinct factors, and (2') compute the number, $P'(n)$, of distinct representations of a number n as a sum of the form $n = a_j + a_k + \cdots$, $a_i \in S$, that is, of distinct terms. The generating functions for the sequences R, P, R', and P' are given by

$$\prod (1 - a_i^{-s})^{-1} = \sum R(n) n^{-s} ,$$

$$\prod (1 - x^{a_i})^{-1} = \sum P(n) x^n ,$$

$$\prod (1 + a_i^{-s}) = \sum R'(n) n^{-s} ,$$

$$\prod (1 + x^{a_i}) = \sum P'(n) x^n .$$

The reciprocals of these generating functions are also of interest. For example, those for R and P lead to generalizations of the Möbius function ($\mu(n)$ is obtained for $a_i = p_i$) and of the Euler identity (whis is obtained for $a_i = i$), respectively,

$$\prod (1 - a_i^{-s}) = \sum M(n)n^{-s} ,$$

$$\prod (1 - x^{a_i}) = \sum K(n)x^n .$$

2. REPRESENTATIONS AS A PRODUCT

From the generating function we can develop a "sieve" technique ("sieve" used in the sense given in [1]) for computing the values of the sequence R. The generating function is rewritten in the form

$$\prod(1 - a_n^{-s})^{-1} = \prod(1 + a_n^{-s} + a_n^{-2s} + \cdots) ,$$

where the products extend over $a_n \in S$. We need to know the sequence $\{a_n\}$ for $1 \leq n \leq N$ in advance as the input, if we are to compute $R(n)$ for $1 \leq n \leq N$. Table 1 illustrates this process for the input sequence of Fibonacci numbers 2, 3, 5, 8, $\cdots$.

Table 1
Product Representations, $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$R^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R^{(1)}$	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$R^{(2)}$	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1
$R^{(3)}$	1	1	1	1	1	1	0	1	1	1	0	1	0	0	1	1	0	1	0	1	0	0	0	1	1	0	1	0	0	1	0	1
$R^{(4)}$	1	1	1	1	1	1	0	2	1	1	0	1	0	0	1	2	0	1	0	1	0	0	0	2	1	0	1	0	0	1	0	2
$\cdots$	$\cdots$																															
R	1	1	1	1	1	1	0	2	1	1	0	1	1	0	1	2	0	1	0	1	1	0	0	2	1	1	1	0	0	1	0	2

To begin, let $R^{(0)}$ denote the sequence with 1 in position 1 and 0 in position n for $2 \leq n \leq N$. In order to obtain the sequence $R^{(1)}$ we bring down 1 into position 1 and then add the entry from position 1 of $R^{(1)}$ to the entry at position $1a_1$ of $R^{(0)}$ and enter the sum in position a_1 of $R^{(1)}$, then the entry from position 2 of $R^{(1)}$ is added to the entry in position $2a_1$ of $R^{(0)}$ and the sum is entered in position $2a_1$ of $R^{(1)}$, etc. At position $n \neq ma_1$ of $R^{(1)}$ we simply use the entry from position n of $R^{(0)}$. This set of subprocesses using a_1 is stopped when $ma_1 > N$. To continue, the entry from position 1 of $R^{(1)}$ is entered in position 1 of $R^{(2)}$ and in general the entry in position n of $R^{(1)}$ is computed successively for $1 < n \leq N$ by either entering in position n of $R^{(2)}$ the entry from position n of $R^{(1)}$ if $n \neq ma_2$, but if $n = ma_2$ by adding the entry at position m of $R^{(2)}$ to the entry at position ma_2 of $R^{(1)}$ and entering the sum in position n of $R^{(2)}$. For the general iterative step going from $R^{(k-1)}$ to $R^{(k)}$, since $R^{(k)}(n)$ sometimes depends on previous entries in $R^{(k)}$, we use these formulas sequentially for $n = 1, 2, 3, \cdots$,

$$R^{(k)}(n) = R^{(k-1)}(n) \qquad \text{if} \quad a_k \nmid n ,$$

$$R^{(k)}(n) = R^{(k-1)}(n) + R^{(k)}(n/a_k) \qquad \text{if} \quad a_k \mid n .$$

We stop each subprocess when $ma_k > N$ and we stop the entire process when $a_k > N$; the result is the sequence of values of R for $1 \leq n \leq N$. It should be noted that the sequence $R^{(k-1)}$ can be destroyed entry-by-entry as $R^{(k)}$ is generated.

The reasoning behind the workings of the process is as follows. Suppose that we have already generated the sequence $R^{(k-1)}$. We want to multiply that series generated by the function

$$\prod_{n=1}^{k-1} (1 - a_n^{-s})^{-1}$$

by the series

$$(1 - a_k^{-s})^{-1} = 1 + a_k^{-s} + a_k^{-2s} + \cdots .$$

This is equivalent to the generation of $R^{(k)}$ from $R^{(k-1)}$. Actually, this multiplication is equivalent to successively expanding the scale for $R^{(k-1)}$ by a factor a_k and adding the expanded result to the sequence $R^{(k-1)}$ itself. This is illustrated in Table 2 in which $R^{(k-1)}(n)$ is denoted by c_n.

Table 2
Product Pattern

c_1	c_2	$\cdots$	c_m	$\cdots$	c_{a_k}	$\cdots$	c_{2a_k}	$\cdots$	c_{ma_k}	$\cdots$	$c_{a_k^2}$	$\cdots$	$c_{ma_k^2}$	$\cdots$
					c_1		c_2	$\cdots$	c_m	$\cdots$	c_{a_k}	$\cdots$	c_{ma_k}	$\cdots$
											c_1	$\cdots$	c_m	$\cdots$
														$\cdots$

From the diagram in Table 2 we note that rows, beginning with the second, are repeats of the first row, but with the scale expanded successively by the factor a_k. If we consider a column which is to be summed, we note that the quantity to be added to c_{na_k} is merely the sum appearing in the column headed by c_n, hence we obtain the iteration equations.

In the intermediate steps the number $R^{(k)}(n)$ has an interpretation as the number of representations of n as a product using only elements of the finite subsequence $a_1, a_2, \cdots, a_k$.

3. REPRESENTATIONS AS A SUM (PARTITIONS)

Here we have a quite closely analogous case to that of the previous section. The generating function is rewritten

$$\Pi\left(1 - x^{a_n}\right)^{-1} = \Pi\left(1 + x^{a_n} + x^{2a_n} + \cdots\right) .$$

In Table 3 the operations are diagrammed analogous to Table 2 except that here we notice that the rows are shifted by an amount a_k and then successively added to the $(k - 1)$-sequence.

Table 3
Sum Pattern

$$\begin{array}{cccccccccccccc}
c_0 & c_1 & c_2 & \cdots & c_{a_k} & c_{a_k+1} & \cdots & c_{2a_k} & \cdots & c_{3a_k} & \cdots & c_{ma_k} & & \cdots \\
 & & & & c_0 & c_1 & \cdots & c_{a_k} & \cdots & c_{2a_k} & \cdots & c_{(m-1)a_k} & & \cdots \\
 & & & & & & & c_0 & \cdots & c_{a_k} & \cdots & c_{(m-2)a_k} & & \cdots \\
 & & & & & & & & & c_0 & \cdots & & & \\
 & & & & & & & & & & \cdots & & & \\
 & & & & & & & & & & & c_0 & &
\end{array}$$

(**Here** the sequence is indexed from 0.) Reading the columns as in Table 2 we have the interation formulas

$$P^{(k)}(n) = P^{(k-1)}(n) \qquad \text{if } a_k \nleq n ,$$

$$P^{(k)}(n) = P^{(k-1)}(n) + P^{(k)}(n - a_k) \qquad \text{if } a_k \leq n .$$

The initial sequence $P^{(0)}$ corresponds to $R^{(0)}$; a 1 appears in position 0 and 0 appears otherwise. The process is stopped similarly. It is of interest to note that $a_k \leq n$ here replaces $a_k \mid n$ and that $n - a_k$ replaces n/a_k. In Table 4, the case of the Fibonacci sequence 1, 2, 3, 5, 8, $\cdots$ is illustrated.

Table 4
Sum Representations, $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$P^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$P^{(2)}$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11
$P^{(3)}$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48
$P^{(4)}$	1	1	2	3	4	6	8	10	13	16	20	24	29	34	40	47	54	62	71	80	91	102
$\cdots$	$\cdots$																					
P	1	1	2	3	4	6	8	10	14	17	22	27	33	41	49	59	71	83	99	115	134	157

In the intermediate steps the number $P^{(k)}(n)$ denotes the number of partitions of n involving only those numbers $a_1, a_2, \cdots, a_k$.

4. REPRESENTATIONS WITH DISTINCT ELEMENTS

If it is required that the representations involve no repeated elements, that is, "squarefree" products or "pairfree" sums, the generating functions are simpler and hence the computational process is also simpler. By reasoning somewhat analogous to the previous two sections we obtain the recurrence formulas

$$\begin{aligned} R'^{(k)}(n) &= R'^{(k-1)}(n) && \text{if } a_k \nmid n, \\ R'^{(k)}(n) &= R'^{(k-1)}(n) + R'^{(k-1)}(n/a_k) && \text{if } a_k \mid n, \end{aligned}$$

where we start with $R'^{(0)}(1) = 1$ and $R'^{(0)}(n) = 0$ for $n > 1$. The case of distinct terms of a sum is quite analogous to this. Starting with $P'^{(0)}(0) = 1$, $P'^{(0)}(n) = 0$ for $n > 0$ we obtain the recurrences

$$\begin{aligned} P'^{(k)}(n) &= P'^{(k-1)}(n) && \text{if } a_k \nleq n, \\ P'^{(k)}(n) &= P'^{(k-1)}(n) + P'^{(k-1)}(n - a_k) && \text{if } a_k \le n. \end{aligned}$$

The only alteration required to change the formulas of the previous sections to these is the change in the upper index on the second term. Tables 5 and 6 display the computations of R' and P' for the Fibonacci sequence.

Table 5
Squarefree Products, $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$R'^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(1)}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(2)}$	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(3)}$	1	1	1	0	1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
...	...																															
R	1	1	1	0	1	1	0	1	0	1	0	0	1	0	1	1	0	0	0	0	1	0	0	1	0	1	0	0	0	1	0	0

Table 6
Pairfree Sums, $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$P'^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P'^{(1)}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(Continued on following page.)

Table 6 (Continued

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$P'^{(2)}$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P'^{(3)}$	1	1	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P'^{(4)}$	1	1	1	2	1	2	2	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P'^{(5)}$	1	1	1	2	1	2	2	1	3	2	2	3	1	2	2	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
...	...																															
P	1	1	1	2	1	2	2	1	3	2	2	3	1	3	3	2	4	2	3	3	1	4	3	3	5	2	4	4	2	5	3	3

5. OTHER FORMS

As was remarked in the first section, the reciprocals of the generating functions are also of interest. Tables 7 and 8 illustrate these analogs for the Fibonacci sequence.

Table 7
Product "Reciprocal," $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$M^{(0)}$	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(1)}$	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(2)}$	+1	-1	-1	0	0	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(3)}$	+1	-1	-1	0	-1	+1	0	0	0	+1	0	0	0	0	+1	0	0	0	0	0	0
...	...																				
M	+1	-1	-1	0	-1	+1	0	-1	0	+1	0	0	-1	0	+1	+1	0	0	0	0	-1

Table 8
Sum "Reciprocal," $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$K^{(0)}$	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(1)}$	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(2)}$	+1	-1	-1	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(3)}$	+1	-1	-1	0	+1	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(4)}$	+1	-1	-1	0	+1	0	0	+1	0	-1	-1	+1	0	0	0	0	0	0	0	0	0	0
...	...																					
K	+1	-1	-1	0	+1	0	0	+1	-1	0	0	+1	-1	-1	+1	0	0	0	+1	-1	-1	0

Note that the only alterations required in the techniques used involves the changes of signs; this is equivalent to switching the primed and unprimed formulas and subtracting the second terms instead of adding these terms.

All of the cases which have been considered are special cases of the general formulas

$$\prod(1 + f(a_k)a_k^{-s} + f(a_k^2)a^{-2s} + \ldots) = \sum F(n)n^{-s},$$

$$\prod(1 + g(a_k)x^{a_k} + g(a_k^2)x^{2a_k} + \cdots) = \sum G(n)x^n.$$

Other cases can certainly be derived and similar lines of reasoning can be carried out for the simpler cases. More complicated cases can also be worked out, if the process is generalized somewhat, but they become messy.

REFERENCES

1. R. B. Buschman, "Some Simple Sieves," Fibonacci Quarterly, Vol. 11, No. 3, pp. 247-254.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 3rd Ed., Oxford, Clarendon Press, 1954.

ERRATA

Please make the following changes in the article, "A New Look at Fibonacci Generalizations," by N. T. Gridgeman, appearing in Vol. 11, No. 1, pp. 40-55.

Page 40, Eqs. (1) and (2). Please insert an opening bracket immediately following the summation sign, and a closing bracket immediately following "B" in both cases. In Eq. (2), please change the lower limit of the summation to read: "i=0" instead of "m=0."

Page 41, Table 1. Please add continue signs, i.e., $\vdots \quad \vdots$, at the end of the table.

Page 42, line 14 from bottom: Please correct spelling from "superflulous" to "superfluous."

Page 42, line 7 from bottom: Please insert a space between "over" and "positive."

Page 44, line 10: Please change "member" to read "members."

Page 45, Eq. (11): Please add an opening bracket immediately before $\binom{h}{g-1}$ and a closing bracket at the end of the line.

Page 46, line 10: Please change the first fraction to read " $\sqrt{9}/2 = 2;$"

Page 47, Eq. (18): Please correct the numerator to read:

$$[N - (B - 1)(1/2 - R)](1/2 + R)^n - [N - (B - 1)(1/2 + R)](1/2 - R)^n$$

[Continued on page 306.]