# AN INEQUALITY IN A CERTAIN DIOPHANTINE EQUATION

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The Diophantine Equation

(1) $$x_1^p + \cdots + x_n^p = y^p ,$$

where p is an odd prime number $>1$, $n \geq 2$ and $1 \leq x_1 \leq \cdots \leq x_n$, is known to possess general solutions for $n = 2$, $p = 2$; $n = 3$, $p = 3$; for other values of n and p, no general solutions are known, although computer searches for solutions of such equations can easily be carried through by assigning a value to y and n and then allowing the corresponding x's to take all values from $x_1 = \cdots = x_n = 1$ to $x_1 = \cdots = x_n = y - 1$; in each case, different primes p are tested to see whether Eq. (1) is satisfied. The labor involved, however, is drastically reduced by realizing that for a given n and y possible primes p which can satisfy Eq. (1), have an upper bound, above which no solutions are possible. This statement is a consequence of examining properties of the function $\psi$ which is defined by

(2) $$\psi = (x_1 + \cdots + x_n)/y ,$$

where y is given by Eq. (1), subject to the restrictions stated above. The relevant property of $\psi$ is given by the following:

Theorem. The function $\psi$ is bounded above by $n^{1-\frac{1}{p}}$ and below by $(1 + 2p/y)$ or 1 depending on whether the solution to Eq. (1) are integers or not, respectively.

Proof of Theorem. From elementary calculus

$$d\psi = \sum_{i=1}^{m} \frac{\partial\psi}{\partial x_i} dx_i$$

and $\psi$ has a turning point when

$$\frac{\partial \psi}{\partial x_i} = 0 \qquad (1 \leq i \leq n) .$$

The conditions

$$\frac{\partial \psi}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x_2} = 0 \qquad (i \neq 1)$$

result in the equations

(3) $$x_1^p + \cdots + x_i^p + \cdots + x_n^p - x_1^{p-1}(x_1 + \cdots + x_i + \cdots + x_n) = 0$$

(4) $$x_1^p + \cdots + x_i^p + \cdots + x_n^p - x_i^{p-1}(x_1 + \cdots + x_i + \cdots + x_n) = 0 .$$

Subtracting Eq. (3) from Eq. (4) gives the condition for a turning point as

(5) $$x_1 = x_2 = \cdots = x_n .$$

From which we deduce that $\max(\psi) = n^{1-\frac{1}{p}}$. The lower bound of $x$ depends on whether the $x$'s are restricted to integers. Thus if the $x$'s are non-integral, then we note that $y^p = x_1^p + \cdots + x_n^p < (x_1 + \cdots + x_n)^p$ so that $1 < \psi$. If the $x$'s are integers only, we use the little Fermat theorem $x_i^p \equiv x_i \pmod{p}$. But since $(x_i^p - x_i)$ is even and $p$ is odd by hypothesis, it follows that $x_i^p \equiv x_i \pmod{2p}$ and hence using Eq. (1) we deduce that $x_1 + \cdots + x_m \equiv y \pmod{2p}$ from which it follows immediately that $y + 2p \le x_1 + \cdots + x_n$. The case of $p = 2$ deserves special attention. Using the same reasoning as above, we obtain the inequality:

(6) $$y + 2 \le x_1 + \cdots + x_n \le \sqrt{n}\, y .$$

Moreover, it is easy to derive solutions for the equation

$$\sum_{i=1}^{m} x_i^2 = y^2$$

for any $n$ by using the well known general solution for $n = 2$ — i.e., the identity $(2ab)^2 + (a^2 - b^2)^2 = (a^2 + b^2)^2$. Thus putting $a = n$, and $b = n + 1$, we obtain:

$$(2n + 1)^2 + (2n(n + 1))^2 + (2n(n + 1) + 1)^2 .$$

Now putting $n = m(m + 1)$ and using Eq. (6) gives:

(7) $$(2m + 1)^2 + (2U)^2 + (2U(U + 1))^2 = (2U(U + 1) + 1)^2 ,$$

with $U = m(m + 1)$. It is easy to use induction to show that this method gives an identity in $m$. We may write $m = a/b$ and multiply throughout by $b^{2^n}$ to obtain an identity in $a$ and $b$.