ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Each proposed problem or solution should be submitted on a separate sheet or sheets, preferably typed in double spacing, in the format used below, to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131.

Solutions should be received within four months of the publication date of the proposed problem.

DEFINITIONS

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n;\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-262 Proposed by Herta T. Freitag, Roanoke, Virginia

(a) Prove that the sum of n consecutive Lucas numbers is divisible by 5 if and only if n is a multiple of 4.

(b) Determine the conditions under which a sum of n consecutive Lucas numbers is a multiple of 10.

B-263 Proposed by Timothy B. Carroll, Graduate Student, Western Michigan University, Kalamazoo, Michigan.

Let $S_n = a^n + b^n + c^n + d^n$ where $a, b, c,$ and d are the roots of $x^4 - x^3 - 2x^2 + x + 1 = 0$.

(a) Find a recursion formula for S_n.

(b) Express S_n in terms of the Lucas number L_n.

B-264 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Use the identities $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and $F_n^2 - F_{n-2}F_{n+2} = (-1)^n$ to obtain a factorization of $F_n^4 - 1$.

B-265 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois

Let F_n and L_n be designated as $F(n)$ and $L(n)$. Prove that

$$F(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1].$$

B-266 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois

Let L_n be designated as $L(n)$. Prove that

$$L(3^n) = \prod_{k=0}^{n-1} [L(2 \cdot 3^k) + 1] \quad .$$

B-267 Proposed by Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, California.

Let a regular pentagon of side p, a regular decagon of side d, and a regular hexagon of side h be inscribed in the same circle. Prove that these lengths could be used to form a right triangle; i.e., that $p^2 = d^2 + h^2$.

SOLUTIONS

OF THREE, WHO IS SHE?

B-238 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Can you guess WHO IS SHE? This is an easy simple addition and SHE is divisible by 29.

$$\begin{array}{r} \text{WHO} \\ \underline{\text{IS}} \\ \text{SHE} \end{array}$$

Solution by John W. Milsom, Butler County Community College, Butler, Pennsylvania.

Although it is not stated in the problem, assume (as is customary) that distinct letters represent distinct digits.

The letter I must be replaced with the number 9 (base 10). W is one less than S. Examining the three-digit numbers which are divisible by 29, there are three sets of numbers which satisfy the conditions imposed by the problem.

$$\begin{array}{rrrr} \text{WHO} & 628 & 714 & 743 \\ \underline{\text{IS}} & \underline{97} & \underline{98} & \underline{98} \\ \text{SHE} & 725 & 812 & 841 \end{array}$$

Thus WHO IS SHE can be

1.	628	97	725
2.	714	98	812
3.	743	98	841 .

Also solved by Harold Don Allen, Paul S. Bruckman, J. A. H. Hunter, Robert Kaplar, Jr., Edgar Karst, David Zeitlin, and the Proposer. At least one of the solutions was found by Kim Bachick, Warren Cheves, and Herta T. Freitag. A solution with W = E was found by Richard W. Sielaff.

INEQUALITY ON GENERALIZED BINOMIALS

B-239 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let $p > 0$, $q > 0$, $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = pu_n + qu_{n-1}$ $(n \geq 1)$. Put

$$\left\{ {n \atop k} \right\} = u_n u_{n-1} \cdots u_{n-k+1} / u_1 u_2 \cdots u_k, \qquad \left\{ {n \atop 0} \right\} = 1 .$$

Show that

$$(*) \qquad \left\{ {n \atop k} \right\}^2 - p^2 \left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} > 0 \qquad (0 \leq k \leq n) .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Let

$$L = \left\{ {n \atop k} \right\}^2 - p^2 \left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} .$$

When $n = 2$ and $k = 1$ the inequality is not strict since $L = 0$. Now

$$L = (u_{n-k+1} u_{k+1} - p^2 u_{n-k} u_k) u_n^2 \cdots u_{n-k+2}^2 u_{n-k+1} / u_1^2 \cdots u_k^2 u_{k+1} .$$

Also

$$\begin{aligned} u_{n-k+1} u_{k+1} - p^2 u_{n-k} u_k &= (pu_{n-k} + qu_{n-k-1})(pu_k + qu_{k-1}) - p^2 u_{n-k} u_k \\ &= pq(u_{n-k} u_{k-1} + u_{n-k-1} u_k) + q^2 u_{n-k-1} u_{k-1} . \end{aligned}$$

Since u_k for $k > 0$, p, and q are positive, L is a product of positive numbers except for $n = 2$, $k = 1$.

Also solved by H. W. Gould and the Proposer.

THE MISSING LUCAS FACTOR

B-240 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California.

Prove that, for all positive integers n, $3F_{n+2}F_{n+3}$ is an exact divisor of

$$7F_{n+2}^3 - F_{n+1}^3 - F_n^3 .$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let F_m be denoted by a, b, c, and d when m is n, $n+1$, $n+2$, and $n+3$, respectively. Let $E = 7c^3 - b^3 - a^3$. Then $E = 7c^3 - (d-c)^3 - (2c-d)^3 = 3cd(3c-d)$ and so $(3cd) \mid E$ as desired. (One may note that the remaining factor $3c - d$ equals L_{n+1}.)

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, L. Carlitz, Warren Cheves, Herta T. Freitag, J. A. H. Hunter, Edgar Karst, Graham Lord, F. D. Parker, David Zeitlin, and the Proposer.

THREE FACES OF A POSSIBLE PRIME

B-241 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

If $2F_{2n-1}F_{2n+1} - 1$ and $2F_{2n}^2 + 1$ are both prime numbers, then prove that

$$F_{2n}^2 + F_{2n-1}F_{2n+1}$$

is also a prime number.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Since

$$F_{2n+1}F_{2n-1} - F_{2n}^2 = 1, \quad 2F_{2n-1}F_{2n+1} - 1 = 2F_{2n}^2 + 1 = F_{2n}^2 + F_{2n-1}F_{2n+1}.$$

If any one of these three equal expressions represents a prime, so do the other two.

Also solved by James D. Bryant, Edgar Karst, David Zeitlin, and the Proposer.

FIBONACCI-PASCAL PROPORTION

B-242 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n}{k-1} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k < n$).

Solution by the Proposer.

Let

$$R = \binom{n}{k} \div \binom{n}{k-1}.$$

Then

$$R = [n!/k!(n-k)!][(k-1)!(n-k+1)!/n!] = (n-k+1)/k.$$

Then we can make R equal to F_m/F_{m+1} by choosing k as tF_{m+1} and n as $tF_{m+2} - 1$, with t any positive integer.

ANOTHER ELUSIVE PLEASING PROPORTION

B-243 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n+1}{k} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k \le n$).

Solution by the Proposer.

Here the given ratio of binomial coefficients equals $(n-k+1)/(n+1)$ and this becomes F_m/F_{m+1} when $n = tF_{m+1} - 1$ and $k = tF_{m-1}$, with t any positive integer.