RECURSION - TYPE FORMULAE FOR PARTITIONS INTO DISTINCT PARTS

DEAN R. HICKERSON
University of California, Davis

A recursion formula for $p(n)$, the number of partitions of n, is given by the Euler identity

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ &\qquad + p(n-12) + p(n-15) - - + + \cdots \\ &= \sum_{i \neq 0} (-1)^{i+1}\, p\left(n - \tfrac{1}{2}(3i^2 + i)\right) , \end{aligned} \tag{1}$$

where the sum extends over all integers i, except $i = 0$, for which the arguments of the partition function are nonnegative (see [1]).

This paper presents a recursion type formula for $q(n)$, the number of partitions of n into distinct parts, in terms of $p(k)$ for certain $k \leq n$. In addition a recursion type formula is presented for $q(a,m,n)$, the number of partitions of n into distinct parts congruent to $\pm a \pmod m$, in terms of $p(k)$ for certain $k \leq n$.

Theorem 1. If $n \geq 0$, and $q(n)$ is the number of partitions of n into distinct parts, then

$$q(n) = \sum_{i=-\infty}^{\infty} (-1)^i p(n - (3i^2 + i)) , \tag{2}$$

where the sum extends over all integers i for which the arguments of the partition function are nonnegative.

Proof. We have

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i=0}^{\infty} (1 + x^i) = \prod_{j=0}^{\infty} (1 - x^j)^{-1}$$

$$\cdot \prod_{i=0}^{\infty} \{(1 - x^i)(1 + x^i)\} = \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \prod_{i=0}^{\infty} (1 - (x^2)^i)$$

$$= \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \left(\sum_{i=-\infty}^{\infty} (-1)^i (x^2)^{\frac{3i^2+i}{2}}\right) = \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \left(\sum_{i=-\infty}^{\infty} (-1)^i x^{3i^2+i}\right),$$

and the result follows by equating coefficients of x^n on both sides of this equation.

Corollary. If $n \geq 0$, then

$$q(n) = p(n) + \sum_{i=1}^{\infty} (-1)^i \{p(n - (3i^2 - i)) + p(n - (3i^2 + i))\}$$

$$= p(n) - p(n-2) - p(n-4) + p(n-10) + p(n-14) - p(n-24)$$

$$- p(n-30) + + - - \cdots .$$

Proof. This follows from Eq. (2) by rearranging the right-hand side.

Theorem 2. If $m \geq 3$, $1 \leq a < m/2$, $n \geq 0$ and $q(a,m,n)$ is the number of partitions of n into distinct parts congruent to $\pm a \pmod{m}$, then

(3) $$q(a,m,n) = \sum_{m|(n+aj)} p\left(\frac{n+aj}{m} - \frac{j^2+j}{2}\right) .$$

Proof. We have

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \prod_{i=0}^{\infty} \{(1 + x^{im+a})(1 + x^{im+m-a})\}$$

$$= \prod_{i=0}^{\infty} (1 - x^{im+m})^{-1} \cdot \prod_{i=0}^{\infty} \{(1 - x^{im+m})(1 + x^{im+a})(1 + x^{im+m-a})\}$$

$$= \left(\sum_{i=0}^{\infty} p(i)x^{im}\right) \cdot \prod_{r=1}^{\infty} \{(1 - x^{rm})(1 + x^{rm+a-m})(1 + x^{rm-a})\} .$$

By Jacobi's identity,

$$\prod_{r=1}^{\infty} \{(1 - q^{2r})(1 + zq^{2r-1})(1 + z^{-1}q^{2r-1})\} = \sum_{j=-\infty}^{\infty} z^j q^{j^2} .$$

with

$$q = x^{\frac{m}{2}} \quad \text{and} \quad z = x^{a-\frac{m}{2}} ,$$

we find

$$\prod_{r=1}^{\infty}\{(1 - x^{rm})(1 + x^{rm+a-m})(1 + x^{rm-a})\} = \sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj} .$$

Therefore,

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \left(\sum_{i=0}^{\infty} p(i)x^{im}\right)\left(\sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj}\right).$$

Since $p(i) = 0$ for $i < 0$, we have

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \left(\sum_{i=-\infty}^{\infty} p(i)x^{im}\right)\left(\sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj}\right).$$

Thus,

$$q(a,m,n) = \sum p\left(\frac{n - \left(m\left(\frac{j^2+j}{2}\right) - aj\right)}{m}\right),$$

where the sum extends over all integral values of j for which

$$\frac{n - \left(m\left(\frac{j^2+j}{2}\right) - aj\right)}{m}$$

is an integer. Clearly, this is an integer if and only if $m \mid (n + aj)$. Therefore,

$$q(a,m,n) = \sum_{m \mid (n+aj)} p\left(\frac{n + aj}{m} - \frac{j^2 + j}{2}\right),$$

as required.

Corollary. Let $m \geq 3$, $1 \leq a < m/2$, and $n \geq 0$. Let

$$a_1 = \frac{a}{(a,m)} \quad \text{and} \quad m_1 = \frac{m}{(a,m)} .$$

If $(a,m) \nmid n$, then $q(a,m,n) = 0$. If $(a,m) \mid n$ and j_0 is some solution of the congruence

$$a_1 j \equiv -\frac{n}{(a,m)} \pmod{m_1},$$

then

$$q(a,m,n) = \sum_{k=-\infty}^{\infty} p\left(\left(\frac{n + aj_0}{m} - \frac{j_0^2 + j_0}{2}\right) - \left(\frac{m_1^2}{2}k^2 + \left(j_0m_1 + \frac{m_1}{2} - a_1\right)k\right)\right).$$

Proof. If $(a,m) \nmid n$, then there are no values of j for which $m \mid (n + aj)$. Therefore, the sum in Theorem 2 is empty and $q(a,m,n) = 0$.

Suppose $(a,m) \mid n$ and

$$a_1j_0 \equiv -\frac{n}{(a,m)} \pmod{m_1}.$$

Then for any integer j, $m \mid (n + aj)$, if and only if $j \equiv j_0 \pmod{m_1}$. By Theorem 2,

$$q(a,m,n) = \sum_{j\equiv j_0 \,(\mathrm{mod}\, m_1)}^{\infty} p\left(\frac{n + aj}{m} - \frac{j^2 + j}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty} p\left(\frac{n + a(j_0 + km_1)}{m} - \frac{(j_0 + km_1)^2 + (j_0 + km_1)}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty} p\left(\left(\frac{n + aj_0}{m} - \frac{j_0^2 + j_0}{2}\right) - \left(\frac{m_1^2}{2}k^2 + \left(j_0m_1 + \frac{m_1}{2} - \frac{am_1}{m}\right)k\right)\right).$$

But

$$\frac{am_1}{m} = \frac{a}{m}\frac{m}{(a,m)} = \frac{a}{(a,m)} = a_1,$$

so the proof is complete.

Example. Let $m = 3$ and $a = 1$. Then $(a,m) = 1$, $a_1 = 1$, and $m_1 = 3$. The congruence for j_0 is $j_0 \equiv -n \pmod 3$.

If $n \equiv 0 \pmod 3$, let $j_0 = 0$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 + \frac{1}{2}k\right)\right) = p\left(\frac{n}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 - \frac{1}{2}k\right)\right)\right.$$

$$\left. + p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 + \frac{1}{2}k\right)\right)\right\} = p\left(\frac{n}{3}\right) + p\left(\frac{n}{3} - 4\right) + p\left(\frac{n}{3} - 5\right)$$

$$+ p\left(\frac{n}{3} - 17\right) + p\left(\frac{n}{3} - 19\right) + p\left(\frac{n}{3} - 39\right) + p\left(\frac{n}{3} - 42\right) + \cdots.$$

If $n \equiv 1 \pmod 3$, let $j_0 = -1$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 - \frac{5}{2}k\right)\right)$$

$$= p\left(\frac{n-1}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 - \frac{5}{2}k\right)\right) + p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 + \frac{5}{2}k\right)\right)\right\}$$

$$= p\left(\frac{n-1}{3}\right) + p\left(\frac{n-1}{3} - 2\right) + p\left(\frac{n-1}{3} - 7\right) + p\left(\frac{n-1}{3} - 13\right)$$

$$+ p\left(\frac{n-1}{3} - 23\right) + p\left(\frac{n-1}{3} - 33\right) + p\left(\frac{n-1}{3} - 48\right) + \cdots .$$

If $n \equiv 2 \pmod 3$, let $j_0 = 1$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 + \frac{7}{2}k\right)\right)$$

$$= p\left(\frac{n-2}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 - \frac{7}{2}k\right)\right) + p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 + \frac{7}{2}k\right)\right)\right\}$$

$$= p\left(\frac{n-2}{3}\right) + p\left(\frac{n-2}{3} - 1\right) + p\left(\frac{n-2}{3} - 8\right) + p\left(\frac{n-2}{3} - 11\right)$$

$$+ p\left(\frac{n-2}{3} - 25\right) + p\left(\frac{n-2}{3} - 30\right) + p\left(\frac{n-2}{3} - 51\right) + \cdots .$$

REFERENCE

1. Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, 3rd ed., John Wiley and Sons, Inc., New York, 1972, pp. 226-227.