ROOTS OF FIBONACCI POLYNOMIALS

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Usually the roots of polynomial equations of degree n become more difficult to find exactly as n increases, and for $n \geq 5$, no general formula can be applied. But, for certain classes of polynomials, the roots can be derived by using hyperbolic trigonometric functions. Here, we solve for the roots of Fibonacci and Lucas polynomials of degree n.

The Fibonacci polynomials $F_n(x)$, defined by

$$F_1(x) = 1, \quad F_2(x) = x, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) ,$$

and the Lucas polynomials $L_n(x)$,

$$L_1(x) = x, \quad L_2(x) = x^2 + 2, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x),$$

have the auxiliary equation

$$Y^2 = xY + 1$$

which arises from the recurrence relation, and which has roots

$$\alpha = \frac{x + \sqrt{x^2 + 4}}{2}, \quad \beta = \frac{x - \sqrt{x^2 + 4}}{2} . \tag{1}$$

It can be shown by mathematical induction that

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n(x) = \alpha^n + \beta^n . \tag{2}$$

The first few Fibonacci and Lucas polynomials are given in Table 1. Observe that, when $x = 1$, $F_n(x) = F_n$ and $L_n(x) = L_n$, the n^{th} Fibonacci and Lucas numbers, respectively, See [1] for an introductory article on Fibonacci polynomials.

Now, we develop formulae for finding the roots of any Fibonacci or Lucas polynomial equation using hyperbolic functions defined by

$$\sinh z = (e^z - e^{-z})/2, \quad \cosh z = (e^z + e^{-z})/2$$

Table 1
Fibonacci and Lucas Polynomials

n	F_n	$L_n(x)$
1	1	x
2	x	$x^2 + 2$
3	$x^2 + 1$	$x^3 + 3x$
4	$x^3 + 2x$	$x^4 + 4x^2 + 2$
5	$x^4 + 3x^2 + 1$	$x^5 + 5x^3 + 5x$
6	$x^5 + 4x^3 + 3x$	$x^6 + 6x^4 + 9x^2 + 2$
7	$x^6 + 5x^4 + 6x^2 + 1$	$x^7 + 7x^5 + 14x^3 + 7x$
8	$x^7 + 6x^5 + 10x^3 + 4x$	$x^8 + 8x^6 + 20x^4 + 16x^2 + 2$
9	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$	$x^9 + 9x^7 + 25x^5 + 30x^3 + 9x$

which satisfy, among many other identities,

$$\cosh^2 z - \sinh^2 z = 1$$

$$\cosh iy = \cos y, \qquad \sinh iy = i \sin y .$$

If we let $x = 2 \sinh z$, then $\sqrt{x^2 + 4} = 2 \cosh z$, and from (1), $\alpha = \cosh z + \sinh z = e^z$ while $\beta = \sinh z - \cosh z = -e^{-z}$. Then,

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{e^{zn} - (-1)^n e^{-nz}}{e^z + e^{-z}}$$

$$L_n(x) = \alpha^n + \beta^n = e^{nz} + (-1)^n e^{-nz} .$$

Thus

$$F_{2n}(x) = \frac{\sinh 2nz}{\cosh z}, \qquad F_{2n+1}(x) = \frac{\cosh (2n+1)z}{\cosh z}$$

(3)

$$L_{2n(x)} = 2 \cosh 2nz, \qquad L_{2n+1}(x) = 2 \sinh (2n+1)z .$$

Now, clearly the polynomial equation equals zero when the corresponding hyperbolic function vanishes. For $z = x + iy$ (see [2], p. 55)

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y .$$

Thus, since for real x, $\sinh x = 0$ if and only if $x = 0$, this implies that the zeroes of $\sinh z$ are those of $\sinh iy = i \sin y$, and the zeroes of $\cosh z$ are the zeroes of $\cosh iy = \cos y$. Thus, we can easily find the z's necessary and sufficient for $F_n(x)$ and $L_n(x)$ to be zero.

Example. $F_{2n}(x) = 0$ implies that $\sinh 2nz = 0$, $\cosh z \neq 0$, so that $\sin 2ny = 0$, $\cos y \neq 0$, so $2ny = k\pi$ and $z = iy$. Thus, $x = \pm 2i \sin k\pi/2n$, $k = 0, 1, 2, \cdots, n-1$. Specifically, the zeroes of $F_6(x)$ are given by $x = \pm 2i \sin k\pi/6$, $k = 0, 1, 2$, so that $x = 0, \pm i, \pm i\sqrt{3}$. As a check, since $F_6(x) = x(x^2+1)(x^2+3)$, we can see that the formula is working.

$F_{2n+1}(x) = 0$ only if $\cosh (2n+1)z = 0$, $\cosh z \neq 0$, or when $\cosh (2n+1)iy = \cos (2n+1)y = 0$, $\cos y \neq 0$. Then, $(2n+1)y = (2k+1)\pi/2$, so that

$$z = iy = \frac{i(2k+1)\pi}{(2n+1)2},$$

so that

$$x = \pm 2i \sin \left(\frac{2k+1}{2n+1}\right) \cdot \frac{\pi}{2}, \qquad k = 0, 1, \cdots, n-1.$$

To summarize, taking $x = 2 \sinh z$ leads to the following solutions:

$$
\begin{array}{lll}
F_{2n}(x) = 0: & x = \pm 2i \sin \dfrac{k\pi}{2n}, & k = 0, 1, \cdots, n-1 \\
F_{2n+1}(x) = 0: & x = \pm 2i \sin \left(\dfrac{2k+1}{2n+1}\right) \cdot \dfrac{\pi}{2}, & k = 0, 1, \cdots, n-1 \\
L_{2n}(x) = 0: & x = \pm 2i \sin \left(\dfrac{2k+1}{2n}\right) \cdot \dfrac{\pi}{2}, & k = 0, 1, \cdots, n-1 \\
L_{2n+1}(x) = 0: & x = \pm 2i \sin \dfrac{k\pi}{2n+1}, & k = 0, 1, \cdots, n-1 .
\end{array}
$$

Compare with Webb and Parberry [3].

Suppose that, on the other hand, we start over again with $x = 2i \cosh z$ so that $\sqrt{x^2+4} = 2i \sinh z$, and $\alpha = ie^{z}$, $\beta = ie^{-z}$. Then, by (2),

$$
\begin{aligned}
F_n(x) &= i^{n-1}\left(\frac{e^{zn} - e^{-zn}}{e^{z} - e^{-z}}\right) = i^{(n-1)} \frac{\sinh nz}{\sinh z} \\
L_n(x) &= e^{nz} + e^{-nz} = 2 \cdot i^{n} \cosh nz \quad .
\end{aligned} \tag{4}
$$

Now this looks better. For the Fibonacci polynomials, $F_n(x) = 0$ when $\sinh nz = 0$, $\sinh z \neq 0$. Since $\sinh nz = 0$ if and only if $\sin ny = 0$ or when $z = iy$, we must have $ny = \pm k\pi$ so that $z = \pm ik\pi/n$. Since $i \cosh iy = i \cos y$, $x = 2i \cosh z = 2i \cos k\pi/n$, $k = 1, 2, \cdots, n-1$.

Now, for the Lucas polynomials, $L_n(x) = \cosh nz = 0$ if and only if $\cos ny = 0$, or when ny is an odd multiple of $\pi/2$, and again $z = iy$, so that $x = 2i \cosh z$ becomes $x = 2i \cos (2k+1)\pi/2n$, $k = 0, 1, \cdots, n-1$.

To summarize, taking $x = 2 \cosh z$ leads to the following solutions:

$$F_n(x) = 0: \quad x = 2i\cos\frac{k\pi}{n} \qquad k = 1, 2, \cdots, n-1$$

$$L_n(x) = 0: \quad x = 2i\cos\frac{(2k+1)\pi}{2n}, \qquad k = 0, 1, \cdots, n-1$$

Actually, there is another way, using $F_{2n}(x) = F_n(x)L_n(x)$. Now, if we can solve $F_m(x) = 0$, then the roots of $L_n(x)$ are those roots of $F_{2n}(x)$ which are not roots of $F_n(x)$. Please note how this agrees with our results:

$$F_{2n}(x) = 0 \quad x = 2i\cos\frac{k\pi}{2n}, \qquad k = 1, 2, \cdots, 2n-1$$

$$F_n(x) = 0 \quad x = 2i\cos\frac{2j\pi}{2n}, \qquad j = 1, 2, \cdots, n-1$$

$$L_n(x) = 0 \quad x = 2i\cos\frac{(2j+1)\pi}{2n}, \qquad j = 0, 1, \cdots, n-1 .$$

Thus the roots separate each other.

REFERENCES

1. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII — An Introduction to Fibonacci Polynomials and Their Divisibility Properties," Fibonacci Quarterly, Vol. 8, No. 4, October, 1970, pp. 407-420. Also available as part of A Primer for the Fibonacci Numbers, The Fibonacci Association, Dec., 1972.
2. Ruel V. Churchill, Complex Variables and Applications, McGraw-Hill, New York, 1960.
3. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5, Dec., 1969, pp. 457-463.

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