A PRIMER FOR THE FIBONACCI NUMBERS: PART XII

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ON REPRESENTATIONS OF INTEGERS USING FIBONACCI NUMBERS

In how many ways may a given positive integer p be written as the sum of distinct Fibonacci numbers, order of the summands not being considered? The Fibonacci numbers are $1, 1, 2, 3, 5, \cdots, F_n, \cdots$, where $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For example, $10 = 8 + 2 = 2 + 3 + 5$ is valid, but $10 = 5 + 5 = 1 + 1 + 8$ would not be valid. The original question is an example of a representation problem, which we do not intend to answer fully here. We will explore representations using the least possible number or the greatest possible number of Fibonacci numbers.

1. THE ZECKENDORF THEOREM

First we prove by mathematical induction a lemma which has immediate application.

Lemma: The number of subsets of the set of the first n integers, subject to the constraint that no two consecutive integers appear in the same subset, is F_{n+2}, $n \geq 0$.

Proof. The theorem holds for $n = 0$, for when we have a set of no integers the only subset is ϕ, the empty set. We thus have one subset and $F_{0+2} = F_2 = 1$.

For $n = 1$, 2 subsets: $\{1\}$, ϕ; $F_{1+2} = F_3 = 2$

$n = 2$, 3 subsets: $\{1\}$, $\{2\}$, ϕ; $F_{2+2} = F_4 = 3$

$n = 3$, 5 subsets: $\{1,3\}$, $\{3\}$, $\{2\}$, $\{1\}$, ϕ; $F_{3+2} = F_5 = 5$.

Assume that the lemma holds for $n \leq k$. Then notice that the subsets formed from the first $(k + 1)$ integers are of two kinds — those containing $(k + 1)$ as an element and those which do not contain $(k + 1)$ as an element. All subsets which contain $(k + 1)$ cannot contain element k and can be formed by adding $(k + 1)$ to each subset, made up of the $(k - 1)$ integers, which satisfies the constraint. By the inductive hypothesis there are F_{k+2} subsets satisfying the constraint and using only the first k integers, and there are F_{k+1} subsets satisfying the constraints and using the first $(k - 1)$ integers. Thus there are precisely

$$F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}$$

subsets satisfying the constraint and using the first $(k + 1)$ integers. The proof is complete by mathematical induction.

Now, for the application. The number of ways in which n boxes can be filled with zeros or ones (every box containing exactly one of those numbers) such that no two "ones" appear in adjacent boxes is F_{n+2}. (To apply the lemma simply number the n boxes.) Since we do not wish to use all zeros (ϕ, the empty set in the lemma) the number of logically useable arrangements is $F_{n+2} - 1$. Now, to use the distinctness of the Fibonacci numbers in our representations, we must omit the initial $F_1 = 1$, so that to the n boxes we assign in order the Fibonacci numbers $F_2, F_3, \cdots, F_{n+1}$. This gives us a binary form for the Fibonacci positional notation. The interpretation to give the "zero" or "one" designation is whether or not one uses that particular Fibonacci number in the given representation. If a one appears in the box allocated to F_k, then F_k is used in this particular representation. Notice that since no two adjacent boxes can each contain a "one," no two consecutive Fibonacci numbers may occur in the same representation.

Since the following are easily established identities,

$$F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1 ,$$
$$F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2} - 1 ,$$

using the Fibonacci positional notation the largest number representable under the constraint with our n boxes is $F_{n+2} - 1$. Also the number F_{n+1} is in the n^{th} box, so we must be able to represent at most $F_{n+2} - 1$ distinct numbers with $F_2, F_3, \cdots, F_{n+1}$ subject to the constraint that no two adjacent Fibonacci numbers are used. Since there are $F_{n+2} - 1$ different ways to distribute ones and zeros in our n boxes, there are $F_{n+2} - 1$ different representations which could represent possibly $F_{n+2} - 1$ different integers. That each integer p has a unique representation is the Zeckendorf Theorem [1]:

Theorem. Each positive integer p has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in the representation.

We shall defer the proof of this until a later section. Now, a minimal representation of an integer p uses the least possible number of Fibonacci numbers in the sum. If both F_k and F_{k-1} appeared in a representation, they could both be replaced by F_{k+1}, thereby reducing the number of Fibonacci numbers used. It follows that a representation that uses no two consecutive Fibonacci numbers is a minimal representation and a Zeckendorf representation.

2. ENUMERATING POLYNOMIALS

Next, we use enumerating polynomials to establish the existence of at least one minimal representation for each integer.

An enumerating polynomial counts the number of Fibonacci numbers necessary in the representation of each integer p in a given interval $F_m \le p < F_{m+1}$ in the following way. Associated with this interval is a polynomial $P_{m-1}(x)$. A term ax^j belongs to $P_{m-1}(x)$ if in the interval $F_m \le p < F_{m+1}$, there are a integers p whose minimal representation requires j Fibonacci numbers. For example, consider the interval $F_6 = 8 \le p < 13 = F_7$. Here, we can easily determine the minimal representations

$$\begin{aligned} 8 &= 8 \\ 9 &= 1 + 8 \\ 10 &= 2 + 8 \\ 11 &= 3 + 8 \\ 12 &= 1 + 3 + 8 \;. \end{aligned}$$

Thus, $P_5(x) = x^3 + 2x^2 + x$ because one integer required 3 Fibonacci numbers, 3 integers required 2 Fibonacci numbers, and one integer required one Fibonacci number in its minimal representation. We note in passing that all the minimal representations in this interval contain 8 but not 5. We now list the first nine enumerating polynomials.

	$F_m \le p < F_{m+1}$	$P_{m-1}(x)$
$m = 1$	$1 \le p < 1$	$0 = P_0(x)$
$m = 2$	$1 \le p < 2$	$x = P_1(x)$
$m = 3$	$2 \le p < 3$	$x = P_2(x)$
$m = 4$	$3 \le p < 5$	$x^2 + x = P_3(x)$
$m = 5$	$5 \le p < 8$	$2x^2 + x = P_4(x)$
$m = 6$	$8 \le p < 13$	$x^3 + 3x^2 + x = P_5(x)$
$m = 7$	$13 \le p < 21$	$3x^3 + 4x^2 + x = P_6(x)$
$m = 8$	$21 \le p < 34$	$x^4 + 6x^3 + 5x^2 + x = P_7(x)$
$m = 9$	$34 \le p < 55$	$4x^4 + 10x^3 + 6x^2 + x = P_8(x)$

We shall now proceed by mathematical induction to derive a recurrence relation for the enumerating polynomials $P_m(x)$. It is evident from the definitions that an enumerating polynomial for $F_m \le p < F_{m+2}$ is the sum of the enumerating polynomials for $F_m \le p < F_{m+1}$ and $F_{m+1} \le p < F_{m+2}$. Also it will be proved that the minimal representation of any integer p in the interval $F_m \le p < F_{m+1}$ contains F_m but not F_{m-1}. If we added F_{m+2} to each such minimal representation of p in $F_m \le p < F_{m+1}$ we would get a minimal representation of an integer in the interval

$$L_{m+1} = F_m + F_{m+2} \le p < F_{m+1} + F_{m+2} = F_{m+3} \,.$$

Clearly the enumerating polynomial for this interval is $xP_{m-1}(x)$ since each integer p in this interval has one more Fibonacci number in its minimal representation than did the corresponding integer p in the interval $F_m \le p < F_{m+1}$.

Next, the integers p in the interval $F_{m+2} \le p < F_{m+3}$ require an F_{m+2} in this minimal representation while all the numbers in the interval $F_{m+1} \le p < F_{m+2}$ have F_{m+1} in their minimal representation. In each of these minimal representations remove the F_{m+1}

and put in an F_{m+2}. The resulting integer will have a minimal representation with the same number of Fibonacci numbers as was required before. In other words, the enumerating polynomial $P_{m-1}(x)$ is also the enumerating polynomial for

$$F_{m+2} - F_{m+1} + F_{m+1} \le p' < F_{m+2} - F_{m+1} + F_{m+2} = L_{m+1} .$$

Now, the intervals $F_{m+2} \le p' < L_{m+1}$ and $L_{m+1} \le p' < F_{m+3}$ are not overlapping and exhaust the interval $F_{m+2} \le p < F_{m+3}$. Thus, the enumerating polynomial for this interval is

$$P_{m+1}(x) = P_m(x) + xP_{m-1}(x), \qquad P_0(x) = 0, \qquad P_1(x) = x ,$$

which is the required recurrence relation.

Now, to show by mathematical induction that the minimal representation of any integer p in the interval $F_m \le p < F_{m+1}$ contains F_m but not F_{m-1}, re-examine the preceding steps. Each minimal representation in the interval $F_{m+2} \le p < F_{m+3}$ contains F_{m+2} explicitly since we added F_{m+2} to a representation from the interval $F_m \le p < F_{m+1}$ and by the inductive hypothesis those representations did not contain F_{m+1} but all contained F_m. Next, for the representations from $F_{m+1} \le p < F_{m+2}$, all of which used F_{m+1} explicitly by inductive assumption, we removed the F_{m+1} and replaced it by F_{m+2} so that each representation in $F_{m+2} \le p < F_{m+3}$ contains F_{m+2} but not F_{m+1}. Thus, if the integers p in the previous two intervals, namely, $F_m \le p < F_{m+1}$ and $F_{m+1} \le p < F_{m+2}$, had Zeckendorf representations, then the representations of the integers p in the interval $F_{m+2} \le p < F_{m+3}$ are also Zeckendorf representations.

Now, notice that $P_m(1)$ is the sum of the coefficients of $P_m(x)$, or the count of the numbers for which a minimal representation exists in the interval $F_{m+1} \le p < F_{m+2}$. But, $P_m(1) = F_m$ because $P_1(1) = P_2(1) = 1$ and $P_{m+1}(1) = P_m(1) + 1 \cdot P_{m-1}(1)$, so that the two sequences have the same beginning values and the same recursion formula. The number of integers in the interval $F_{m+1} \le p < F_{m+2}$ is $F_{m+2} - F_{m+1} = F_m$, so that every integer is represented. Thus, at least one minimal representation exists for each integer, and we have established Zeckendorf's theorem, that each integer has a unique minimal representation in Fibonacci numbers. Notice that this means that it is possible to express any integer as a sum of distinct Fibonacci numbers. Also, notice that the coefficients of $P_m(x)$ are the summands along the diagonals of Pascal's triangle summing to F_m with increasing powers as one proceeds up the diagonals beginning with x.

3. THE DUAL ZECKENDORF THEOREM

Suppose that, instead of a minimal representation, we wished to write a maximal representation, or, to use as many distinct Fibonacci numbers as possible in a sum to represent an integer. Then, we want no two consecutive Fibonacci numbers to be missing in the representation. Returning to our n non-empty boxes, for this case we wish to fill the boxes with zeros and ones with no two consecutive zeros. Here we consider n ones interposed by at most one zero. Thus, we have boxes to zero or not to zero. These zeros can occur between the left-most one and the next on the right, between any adjacent pair of ones, and on the right of the last one if necessary. Thus, there are precisely 2^n possibilities, or, 2^n maximal representations can be written using n Fibonacci numbers from among $1, 2, 3, 5, \cdots, F_{2n+1}$.

Now, associate with integers p in the interval $F_n - 1 \le p < F_{n+1} - 1$ an enumerating maximal polynomial $P^*_{n-1}(x)$ which has a term ax^j if a of the integers p require j Fibonacci numbers in their maximal representation. For example, in the interval $F_6 - 1 = 7 \le p < 12 = F_7 - 1$, the maximal representations are

$$
\begin{aligned}
7 &= 5 + 2\\
8 &= 5 + 2 + 1\\
9 &= 5 + 3 + 1\\
10 &= 5 + 3 + 2 + 1\\
11 &= 5 + 3 + 2 + 1\ .
\end{aligned}
$$

Thus, $P_5(x) = x^4 + 3x^3 + x^2$ because one integer requires 4 Fibonacci numbers, 3 integers require 3 Fibonacci numbers, and one integer requires 2 Fibonacci numbers in its maximal representation. Notice that all maximal representations above use 5 but none use 8. The first eight enumerating maximal polynomials are:

	$F_m - 1 \le p < F_{m+1} - 1$	$P^*_{m-1}(x)$
$m = 2$	$0 \le p < 1$	$1 = P^*_1(x)$
$m = 3$	$1 \le p < 2$	$x = P^*_2(x)$
$m = 4$	$2 \le p < 4$	$x^2 + x = P^*_3(x)$
$m = 5$	$4 \le p < 7$	$x^3 + 2x^2 = P^*_4(x)$
$m = 6$	$7 \le p < 12$	$x^4 + 3x^3 + x^2 = P^*_5(x)$
$m = 7$	$12 \le p < 20$	$x^5 + 4x^4 + 3x^3 = P^*_6(x)$
$m = 8$	$20 \le p < 33$	$x^6 + 5x^5 + 6x^4 + x^3 = P^*_7(x)$
$m = 9$	$33 \le p < 54$	$x^7 + 6x^6 + 10x^5 + 4x^4 = P^*_8(x)$

As before, we now derive the recurrence relation for the polynomials $P^*_n(x)$.

Lemma. Each maximal representation for integers p in the interval $F_m - 1 \le p < F_{m+1} - 1$ contains explicitly F_{m-1}.

Proof. We can add F_m to each maximal representation in the interval $F_m - 1 \le p < F_{m+1} - 1$ and these numbers fall in the interval

$$2F_m - 1 \le p' < F_{m+2} - 1\ .$$

We can also add F_m to each maximal representation in the interval $F_{m-1} - 1 \le p < F_{m+1} - 1$ and these numbers fall in the interval

$$F_{m+1} - 1 \le p' < 2F_m - 1 .$$

These two intervals are non-overlapping and exhaustive of the interval

$$F_{m+1} - 1 \le p < F_{m+2} - 1 .$$

Thus, each maximal representation in this interval contains explicitly F_m.

Thus, the enumerating polynomials $P_n^*(x)$ for maximal representations satisfy

$$P_n^*(x) = x\,[P_{n-1}^*(x) + P_{n-2}^*(x)] , \qquad P_1^*(x) = 1, \qquad P_2^*(x) = x ,$$

and again $P_n^*(1) = F_n$. This establishes that each non-negative integer has at least one maximal representation.

Returning to the table of the first eight polynomials $P_n^*(x)$, by laws of polynomial addition, adding the enumerating maximal polynomials yields a count of how many numbers require k Fibonacci numbers in their maximal representation. So, it appears that

$$\begin{aligned}\sum_{n=1}^{\infty} P_n^*(x) &= P_1^*(x) + P_2^*(x) + P_3^*(x) + P_4^*(x) + P_5^*(x) + \cdots + P_n^*(x) + \cdots \\ &= 1 + x + (x^2 + x) + (x^3 + 2x^2) + (x^4 + 3x^3 + x^2) + \cdots \\ &= 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^k x^k + \cdots ,\end{aligned}$$

(That this is indeed the case is proved in the two lemmas following the Dual Zeckendorf Theorem.) In other words, 2^k non-negative integers require k Fibonacci numbers in their maximal representation. But requiring that each integer has at least one maximal representation exhausts the logical possibilities. Thus, each integer has a unique maximal representation in distinct Fibonacci numbers, which proves the Dual Zeckendorf Theorem [2]:

Theorem. Each positive integer has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are omitted in the representation.

Lemma. Let $f_1(x) = 1$, $f_2(x) = x$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$ be the Fibonacci polynomials. Then

$$P_n^*(x^2) = x^{n-1} f_n(x), \qquad n \ge 0 .$$

Proof. We proceed by mathematical induction. Observe that

$$P_1^*(x^2) = 1 = x^0 f_1(x) ,$$

$$P_2^*(x^2) = x^2 = x^1 f_2(x) ,$$

$$P_n^*(x^2) = x^2 [P_{n-1}^*(x^2) + P_{n-2}^*(x^2)] .$$

Assume that

$$P^*_{n-1}(x^2) = x^{n-2} f_{n-1}(x) ,$$
$$P^*_{n-2}(x^2) = x^{n-3} f_{n-2}(x) .$$

Thus

$$\begin{aligned} P^*_n(x^2) &= x^2\left[x^{n-2} f_{n-1}(x) + x^{n-3} f_{n-2}(x)\right] \\ &= x^{n-1}\left[x f_{n-1}(x) + f_{n-2}(x)\right] = x^{n-1} f_n(x) . \end{aligned}$$

Lemma.

$$\sum_{n=1}^{\infty} P^*_n(x) = \frac{1}{1 - 2x} .$$

Proof. The Fibonacci polynomials have the generating function

$$\frac{1}{1 - xt - t^2} = \sum_{n=1}^{\infty} f_n(x)\, t^{n-1} .$$

Now let $x = t$, and then by the previous lemma,

$$\frac{1}{1 - x^2 - x^2} = \sum_{n=1}^{\infty} f_n(x) x^{n-1} = \sum_{n=1}^{\infty} P^*_n(x^2) = \frac{1}{1 - 2x^2} .$$

Therefore,

$$\sum_{n=1}^{\infty} P^*_n(x) = \frac{1}{1 - 2x} = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots .$$

Notice that the polynomials $P^*_n(x)$ have as their coefficients the summands along the rising diagonals of Pascal's triangle whose sums are the Fibonacci numbers but in the reverse order of those for $P_n(x)$. In fact, the minimal enumerating polynomials $P_n(x)$ and the maximal enumerating polynomials $P^*_n(x)$ are related as in the following lemma:

Lemma.

$$P_m(x) = x^m P^*_m(1/x) \quad \text{for } m \geq 1 .$$

Proof. This relationship will be proved by mathematical induction.

$$m = 1: \quad P_1(x) = x = x^1[P^*_1(1/x)]$$

$$m = 2: \quad P_2(x) = x = x^2(1/x) = x^2[P^*_2(1/x)]$$

$$m = 3: \quad P_3(x) = x^2 + x = x^3(1/x + 1/x^2) = x^3[P^*_3(1/x)] .$$

Assume that

$$P_{k-1}(x) = x^{k-1}P^*_{k-1}(1/x) \quad ,$$
$$P_k(x) = x^k P^*_k(1/x) \quad .$$

Then, by the recurrence relations for the polynomials $P_n(x)$ and $P^*_n(x)$,

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + xP_{k-1}(x) \\ &= x^k P^*_k(1/x) + xx^{k-1} P^*_{k-1}(1/x) \\ &= x^{k+1}(1/x)[\, P^*_k(1/x) + P^*_{k-1}(1/x)\,] \\ &= x^{k+1} P^*_{k+1}(1/x) \quad , \end{aligned}$$

which establishes the lemma by mathematical induction.

Then, both the minimal and maximal representations of an integer are unique. Then, an integer has a unique representation in Fibonacci numbers if and only if its minimal and maximal representations are the same, which condition occurs only for the integers of the form $F_n - 1$, $n \geq 3$ [3]. In general, the representation of an integer in Fibonacci numbers is not unique, and, from the above remarks, unless the number is one less than a Fibonacci number, it will have at least two representations in Fibonacci numbers. But, one need not stop here. The Fibonacci numbers F_{2n} and F_{2n+1} can each be written as the sum of distinct Fibonacci numbers $1, 2, 3, 5, 8, \cdots$, in n different ways. For other integers p, the reader is invited to experiment to see what theorems he can produce.

We now turn to representations of integers using Lucas numbers.

4. THE LUCAS CASE

If we change our representative set from Fibonacci numbers to Lucas numbers, we can find minimal and maximal representations of integers as sums of distinct Lucas numbers. The Lucas numbers are $2, 1, 3, 4, 7, 11, \cdots$, defined by $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 1$. (See Brown [6a].)

The derivation of a recursion formula for the enumerating minimal polynomials $Q_n(x)$ for Lucas numbers is very similar to that for the polynomials $P_n(x)$ for Fibonacci numbers. Details of the proofs are omitted here. Now, for integers p in the interval $L_n \leq p < L_{n+1}$, the enumerating minimal polynomial $Q_{n-1}(x)$ has a term dx^j if $\underline{d}$ of the integers p require $\underline{j}$ Lucas numbers in their minimal representation. For example, the minimal representation in Lucas numbers for integers p in the interval $11 = L_5 \leq p < L_6 = 18$ are:

$$\begin{aligned} 11 &= 11 \\ 12 &= 11 + 1 \\ 13 &= 11 + 2 \\ 14 &= 11 + 3 \end{aligned} \qquad \begin{aligned} 15 &= 11 + 4 \\ 16 &= 11 + 4 + 1 \\ 17 &= 11 + 4 + 2 \end{aligned}$$

so that $Q_4(x) = 2x^3 + 4x^2 + x$ since 2 integers require 3 Lucas numbers, 4 integers require 2 Lucas numbers, and one integer requires one Lucas number. Notice that $L_5 = 11$ is included in each representation, but that $L_4 = 7$ does not appear in any representation in this inverval. Also notice that we could have written $16 = 11 + 3 + 2$. To make the minimal representation unique, it is necessary to avoid one of the combinations $L_0 + L_1$ or $L_1 + L_3$: we agree not to use the combination $L_0 + L_2 = 2 + 3$ in any minimal representation unless one or both of L_1 and L_3 also appear. The first nine Lucas enumerating minimal polynomials follow.

	$L_m \le p < L_{m+1}$	$Q_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q_0(x)$
$m = 2$	$3 \le p < 4$	$x = Q_1(x)$
$m = 3$	$4 \le p < 7$	$2x^2 + x = Q_2(x)$
$m = 4$	$7 \le p < 11$	$3x^2 + x = Q_3(x)$
$m = 5$	$11 \le p < 18$	$2x^3 + 4x^2 + x = Q_4(x)$
$m = 6$	$18 \le p < 29$	$5x^3 + 5x^2 + x = Q_5(x)$
$m = 7$	$29 \le p < 47$	$2x^4 + 9x^3 + 6x^2 + x = Q_6(x)$
$m = 8$	$47 \le p < 76$	$7x^4 + 14x^3 + 7x^2 + x = Q_7(x)$
$m = 9$	$76 \le p < 123$	$2x^5 + 16x^4 + 20x^3 + 8x^2 + x = Q_8(x)$

Similarly to $P_n(x)$, by the rules of polynomial addition and because of the way the polynomials $Q_n(x)$ are defined,

$$Q_{n+1}(x) = Q_n(x) + xQ_{n-1}(x), \quad Q_0(x) = 2x, \quad Q_1(x) = x,$$

is the recursion relation satisfied by the polynomials $Q_n(x)$. Here we have the same recursion formula satisfied by the polynomials $P_n(x)$, but with different starting values. Notice that $Q_n(1) = L_n$. As before, $Q_{n-1}(1)$ is the sum of the coefficients of $Q_{n-1}(x)$, or, the count of the numbers for which a minimal representation exists in the interval $L_n \le p < L_{n+1}$, which contains exactly $L_{n+1} - L_n = L_{n-1}$ integers. Thus, each integer has at least one minimal representation in distinct Lucas numbers.

Now, let us reconsider the n boxes. To have a minimal representation, we wish to fill the n boxes with zeros or ones such that no two ones are adjacent and to discard the arrangement using all zeros. As before, there are $F_{n+2} - 1$ such arrangements. Now, establish a Lucas number positional notation by putting the Lucas numbers $L_0, L_1, L_2, L_3, \cdots, L_{n-1}$ into the n boxes. Again, the significance of the ones and zeros is determination of which Lucas numbers are used in the sum. But, notice that $L_0 + L_2 = L_1 + L_3$, which would make more than one minimal representation of an integer possible. To avoid this problem, we consider the first four boxes and reject $L_0 + L_2$ whenever that combination occurs without L_1 or L_2. If such four boxes hold then there are $(n - 4)$ remaining boxes which

0	1	0	1
L_3	L_2	L_1	L_0

can hold F_{n-2} compatible arrangements. Thus, rejecting these endings eliminates F_{n-2} arrangements, making the number of admissible arrangements $F_{n+2} - F_{n-2} - 1 = L_n - 1$. But the Lucas sequence begins with $L_0 = 2$, so that the number L_n is in the box numbered $(n + 1)$. Therefore, using the first n Lucas numbers and the two constraints, we can have at most $L_n - 1$ different numbers represented, for

$$L_1 + L_3 + \cdots + L_{2k-1} = L_{2k} - 2,$$
$$L_2 + L_4 + \cdots + L_{2k-2} = L_{2k-1} - 1,$$

and the $L_0 + L_2$ ending was rejected.

Then, we can have at most $L_n - 1$ different numbers represented using $L_0, L_1, \cdots, L_{n-1}$, but the enumerating minimal polynomial guarantees that each of the numbers 1, 2, 3, $\cdots$, $L_n - 1$, has at least one minimal representation. Thus, the minimal representation of an integer in Lucas numbers, subject to the two constraints given, is unique. This is the Lucas Zeckendorf Theorem.

For the maximal representation of an integer using distinct Lucas numbers, again we will need to use adjacent Lucas numbers whenever possible. In our n boxes, then, we will want to place the ones and zeros so that there never are two consecutive zeros. Also, we need to exclude the ending $L_1 + L_3$ in our representations to exclude the possibility of two maximal representations for an integer, one using $L_0 + L_2 = 5$ and the other $L_1 + L_3 = 5$. We will use the combination $L_1 + L_3$ only when one of L_0 or L_2 occurs in the same maximal representation.

Now, let the enumerating maximal polynomials for the Lucas case for the interval $L_n \le p < L_{n+1}$ be $Q^*_{n-1}(x)$, where dx^j is a term of $Q^*_{n-1}(x)$ if d of the integers p require j Lucas numbers in their maximal representation. For example, the maximal representation in Lucas numbers for integers p in the interval $11 = L_5 \le p < L_6 = 18$ are:

$$\begin{aligned}
11 &= 7 + 3 + 1\\
12 &= 7 + 3 + 2\\
13 &= 7 + 3 + 2 + 1\\
14 &= 7 + 4 + 2 + 1\\
15 &= 7 + 4 + 3 + 1\\
16 &= 7 + 4 + 3 + 2\\
17 &= 7 + 4 + 3 + 2 + 1
\end{aligned}$$

so that $Q^*_4(x) = x^5 + 4x^4 + 2x^3$, since one integer requires 5 Lucas numbers, 4 integers require 4 Lucas numbers, and 2 integers require 3 Lucas numbers in their maximal representation. The first nine polynomials $Q^*_n(x)$ follow.

	$L_m \le p < L_{m+1}$	$Q^*_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q_0^*(x)$
$m = 2$	$3 \le p < 4$	$x^2 = Q_1^*(x)$
$m = 3$	$4 \le p < 7$	$x^3 + 2x^2 = Q_2^*(x)$
$m = 4$	$7 \le p < 11$	$x^4 + 3x^3 = Q_3^*(x)$
$m = 5$	$11 \le p < 18$	$x^5 + 4x^4 + 2x^3 = Q_4^*(x)$
$m = 6$	$18 \le p < 29$	$x^6 + 5x^5 + 5x^4 = Q_5^*(x)$
$m = 7$	$29 \le p < 47$	$x^7 + 6x^6 + 9x^5 + 2x^4 = Q_6^*(x)$
$m = 8$	$47 \le p < 76$	$x^8 + 7x^7 + 14x^6 + 7x^5 = Q_7^*(x)$
$m = 9$	$76 \le p < 123$	$x^9 + 8x^8 + 20x^7 + 16x^6 + 2x^5 = Q_8^*(x)$

The recursion relation for the polynomials $Q_n^*(x)$ can be derived in a similar fashion to $P_n^*(x)$, becoming

$$Q^*_{n+1}(x) = x[\,Q_n^*(x) + Q^*_{n-1}(x)\,], \qquad Q_0^*(x) = 2x, \qquad Q_1^*(x) = x^2 .$$

Notice that the same coefficients occur in the enumerating minimal Lucas polynomial $Q_n(x)$ and in the enumerating maximal Lucas polynomial $Q_n^*(x)$. The relationship in the lemma below could be proved by mathematical induction, paralleling the proof of the similar property of $P_n(x)$ and $P_n^*(x)$ given in the preceding section.

Lemma.

$$Q_m(x) = x^{m+1} Q_m^*(1/x) \qquad \text{for } m \ge 1 .$$

Also, the polynomials $P_n^*(x)$ and $Q_n^*(x)$ are related as follows:

Lemma.

$$Q^*_{n-1}(x) = x P_n^*(x) + x^2 P^*_{n-2}(x), \qquad n \ge 1 ,$$

which could be proved by mathematical induction. Notice that the lemma above becomes the well known identity, $L_{n-1} = F_n + F_{n-2}$, when $x = 1$.

Now we return to our main problem.

By laws of polynomial addition, if we add all polynomials $Q_n^*(x)$, the coefficients in the sum will provide a count of how many integers require k Lucas numbers in their maximal representation. Then, it would appear that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_n^*(x) &= Q_0^*(x) + Q_1^*(x) + Q_2^*(x) + Q_3^*(x) + Q_4^*(x) + \cdots + Q_k^*(x) + \cdots \\ &= 2x + x^2 + (x^3 + 2x^2) + (x^4 + 3x^3) + (x^5 + 4x^4 + 2x^3) + \cdots \\ &= 2x + 3x^2 + 6x^3 + 12x^4 + 24x^5 + \cdots + 3\cdot 2^{k-2} x^k + \cdots ,\end{aligned}$$

so that $3 \cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, $k \geq 2$. A proof that this is the correct computation of the sum of the polynomials $Q_n^*(x)$ follows.

Lemma. If

$$Q_0^*(x) = 2x, \qquad Q_1^*(x) = x^2, \quad \text{and} \quad Q_n^*(x) = x\left[Q_{n-1}^*(x) + Q_{n-2}^*(x)\right],$$

then

$$Q_{n-1}^*(x^2) = x^{n+1}\left[f_n(x) + f_{n-2}(x)\right],$$

where $f_n(x)$ are the Fibonacci polynomials.

Proof. To begin a proof by mathematical induction, observe that

$$\begin{aligned} n = 1: &\quad Q_0^*(x^2) = 2x^2 = x^2(1 + 1) = x^{1+1}[f_1(x) + f_{-1}(x)] \\ n = 2: &\quad Q_1^*(x^2) = x^4 = x^3(x + 0) = x^{2+1}[f_2(x) + f_0(x)] \quad . \end{aligned}$$

Assume that the lemma holds for $(n - 1)$ and $(n - 2)$. Then

$$\begin{aligned} Q_n^*(x^2) &= x^2[Q_{n-1}^*(x^2) + Q_{n-2}^*(x^2)] \\ &= x^2\{x^{n+1}[f_n(x) + f_{n-2}(x)] + x^n[f_{n-1}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}\{[xf_n(x) + f_{n-1}(x)] + [xf_{n-2}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}[f_{n+1}(x) + f_{n-1}(x)], \end{aligned}$$

establishing the lemma by mathematical induction for $n \geq 1$.

Using known generating functions for the Fibonacci polynomials as before,

$$\sum_{n=1}^{\infty} f_n(x)\, t^{n+1} = \frac{t^2}{1 - xt - t^2} \quad ,$$

$$\sum_{n=1}^{\infty} f_{n-2}(x)\, t^{n+1} = \frac{t^2(1 - xt)}{1 - xt - t^2} \quad .$$

Adding,

$$\frac{t^2(2 - xt)}{1 - xt - t^2} = \sum_{n=1}^{\infty} [f_n(x) + f_{n-2}(x)]\, t^{n+1} \quad .$$

Setting $t = x$,

$$\frac{2x^2 - x^4}{1 - 2x^2} = \sum_{n=1}^{\infty} x^{n+1}[f_n(x) + f_{n-2}(x)] = \sum_{n=1}^{\infty} Q_{n-1}^*(x^2) \quad .$$

Therefore,

$$\sum_{n=0}^{\infty} Q_n^*(x) = \frac{2x - x^2}{1 - 2x} = 2x + \frac{3x^2}{1 - 2x} = 2x + \sum_{n=2}^{\infty} 3\cdot 2^{n-2} x^n .$$

To see the reason for the peculiar coefficients $3\cdot 2^{k-1}$, examine the eight possible ways to fill the first four boxes with zeros and ones. Then see how many numbers requiring n Lucas numbers in their maximal representation could be written. In other words, consider how to distribute n ones without allowing two consecutive zeros. The eight cases follow.

L_3	L_2	L_1	L_0	Count of Possibilities $(n \geq 4)$
1	1	1	1	2^{n-4}
1	0	1	0	excluded
0	1	0	1	2^{n-3}
1	1	0	1	2^{n-3}
1	0	1	1	2^{n-3}
1	1	1	0	2^{n-3}
0	1	1	1	2^{n-4}
0	1	1	0	2^{n-3}

Summing the seven useable cases gives

$$5\cdot 2^{n-3} + 2\cdot 2^{n-4} = 6\cdot 2^{n-3} = 3\cdot 2^{n-2}, \qquad n \geq 4,$$

possible maximal representations. The endings with a zero in the left-most box would require that the L_4 box contain a one, while all would have either an L_4 or an L_5 appearing in the representation. The endings listed above do not give the numbers requiring 1, 2, or 3 Lucas numbers in their maximal representation. So, the endings given above do not include the representations of 1 through 9, 11 and 12, which give the first three terms $2x + 3x^2 + 6x^3$ of the enumerating maximal Lucas polynomial sum and explain the irregular first term in the sum of the polynomials $Q_n^*(x)$. The numbers not included in the count of possibilities above follow.

L_4	L_3	L_2	L_1	L_0	representing:	
	0	0	1	0	1	$2x$
	0	0	0	1	2	
	0	0	1	1	3	$3x^2$
	0	1	1	0	4	
	0	1	0	1	5	
	0	1	1	1	6	$6x^3$
	1	0	1	1	7	
	1	1	1	0	8	
	1	1	0	1	9	
1	0	1	1	0	11	
1	0	1	0	1	12	

Now, the enumerating maximal polynomial guarantees that $3 \cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, but examining the possible maximal representations which could be written using k Lucas numbers shows that at most $3 \cdot 2^{k-2}$ different representations could be formed. That is exactly one apiece, so the maximal representation of an integer using Lucas numbers subject to the two constraints, that no two consecutive Lucas numbers are omitted and that the combination $L_3 + L_1$ is not used unless L_0 or L_2 also appear, is unique.

5. CONCLUDING REMARKS

Much interest has been shown in the subject of representations of integers in recent years. Some of the many diverse new results which arise naturally from this paper are recorded here with references for further reading.

That the Fibonacci and Lucas sequences are <u>complete</u> has been shown in this paper, although the property was not named. A sequence of positive integers, $a_1, a_2, \cdots, a_n, \cdots$, is <u>complete</u> with respect to the positive integers if and only if every positive integer m is the sum of a finite number of the members of the sequence, where each member is used at most once in any given representation. (See [4], [5].) For example, the sequence of powers of two is complete; any positive integer can be represented in the binary system of numeration. However, if any power of 2, for example, $1 = 2^0$, is omitted, the new sequence is not complete. It is surprising that, for the Fibonacci sequence where $a_n = F_n$, $n \geq 1$, if any <u>one</u> arbitrary number F_k is missing, the sequence is still complete, but if any two arbitrary Fibonacci numbers F_p and F_q are missing, the sequence is incomplete [4].

The Dual Zeckendorf Theorem has an extension that characterizes the Fibonacci numbers. Brown in [2] proves that, if each positive integer has a unique representation as the sum of distinct members of a given sequence when no two consecutive members of the sequence are omitted in the representation, then the given sequence is the sequence of Fibonacci numbers.

Generalized Fibonacci numbers can be studied in a manner similar to the Lucas case. A set of particularly interesting sequences arising in Pascal's triangle appears in [6]: the sequences formed as the sums of elements of the diagonals of Pascal's left-justified triangle, beginning in the left-most column and going right one and up p throughout the array. (The Fibonacci numbers occur when $p = 1$.) Or, the squares of Fibonacci numbers may be used (see [7]), which gives a complete sequence if members of the sequence can be used twice. Other ways of studying generalized Fibonacci numbers include those given in [8], [9], [10], and [11].

To return to the introduction, Carlitz [12] and Klarner [13] have studied the problem of counting the number of representations possible for a given integer. Tables of the number of representations of integers as sums of distinct elements of the Fibonacci sequence as well as other related tables appear in [14]. The general problem of representations of integers using the Fibonacci numbers, enumerating intervals, and positional binary notation for the representations were given by Ferns [15] while [16] is one of the earliest references following Daykin [8]. The suggested readings and the references given here are by no means exhaustive. The range of representation problems is bounded only by the imagination.

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