POLYNOMIALS ARISING FROM REFLECTIONS ACROSS MULTIPLE PLATES

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1. INTRODUCTION

It is known that reflections of light rays within two glass plates can be expressed in terms of the Fibonacci numbers as mentioned by Moser [1]. Here, we will explore what happens when the number of glass plates is increased. As will be seen, a new set of sequences and polynomials arises.

Assume that one starts with a single light ray and that the surfaces of the glass plates are half-mirrors, such that they both transmit and reflect light. The initial reflection, as a light ray enters the stack of plates, is ignored. Let $P(n,k)$ be the number of possible distinct light paths, where n is the number of reflections and k the number of plates. Figure 1 illustrates the particular case of two glass plates for $n = 0, 1, 2,$ and 3, where we already know that the possible light paths result in the Fibonacci numbers. The dots on the upper surface in this figure indicate the start of a light ray for a distinct possible path for each particular number of reflections.

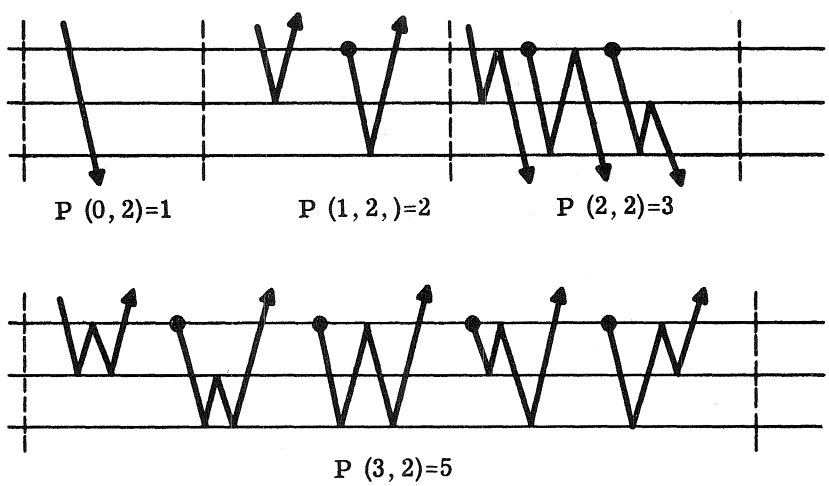

Figure 1

We will now derive a matrix equation which relates the number of distinct reflected paths to the number of reflections and to the number of glass plates and examine a sequence of polynomials arising from the characteristic equations of these matrices.

2. THE $k \times k$ MATRIX Q_k

Consider the bundle of distinct paths along which each light ray has been reflected exactly n times in a collection of k glass plates, as shown in Fig. 2. Let $Q(n,i)$ be the number of rays added by reflection to the bundle at the surface i, $1 \le i \le k$, at which point the rays make the n^{th} reflection. Since the number of rays emerging from the stack of k plates after exactly n reflections is identical to the number of possible distinct light paths for n reflections,

(1) $$P(n,k) \equiv Q(n,k) \quad \text{for all } n \text{ and } k.$$

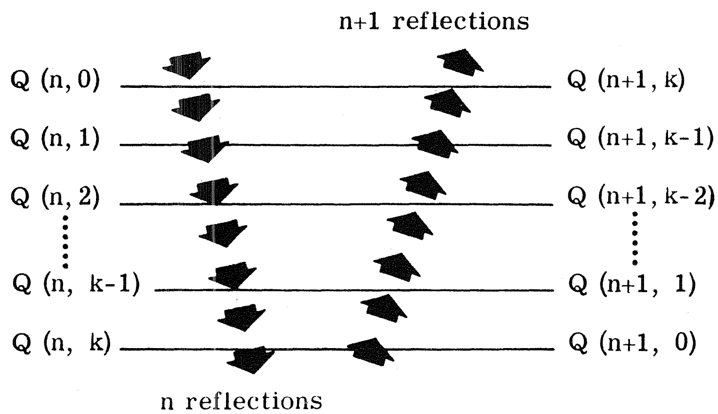

Figure 2

From Figure 2, the following set of equations is then obtained:

$$\begin{aligned}
Q(n+1, k) &= Q(n,k) + Q(n, k-1) + \cdots + Q(n,2) + Q(n,1) \\
Q(n+1, k-1) &= Q(n,k) + Q(n, k-1) + \cdots + Q(n,2) \\
\cdots & \qquad \cdots \\
Q(n+1, 2) &= Q(n,k) + Q(n, k-1) \\
Q(n+1, 1) &= Q(n,k) .
\end{aligned}$$

We can write this set of equations as a matrix equation,

(2) $$\left\| \begin{matrix} Q(n+1, k) \\ Q(n+1, k-1) \\ \cdots \\ Q(n+1, 2) \\ Q(n+1, 1) \end{matrix} \right\| = \left\| \begin{matrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{matrix} \right\| \cdot \left\| \begin{matrix} Q(n,k) \\ Q(n, k-1) \\ \cdots \\ Q(n, 2) \\ Q(n, 1) \end{matrix} \right\|$$

and define Q_k as the square matrix of order k which arises with its elements above and on the minor diagonal all ones and with all zeros below the minor diagonal,

$$Q_k = \begin{Vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{Vmatrix}_{k \times k} \tag{3}$$

Next, we find a recursion relation for the characteristic polynomials for the matrices Q_k. We let

$$D_k(y) = \det (Q_k - yI_k) , \tag{4}$$

where Q_k is given by (3) and I_k is the identity matrix of order k. We display Eq. (4) as

$$D_k(y) = \begin{vmatrix} 1-y & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-y & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1-y & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 0 & \cdots & -y & 0 \\ 1 & 0 & 0 & \cdots & 0 & -y \end{vmatrix}_{k \times k} \tag{5}$$

The determinant on the right side of (5) is now modified by subtracting row 2 from row 1, after which column 2 is subtracted, resulting in (6):

$$\begin{vmatrix} -2y & y & 0 & \cdots & 0 & 1 \\ y & 1-y & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1-y & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 1 & 0 & \cdots & -y & 0 \\ 1 & 0 & 0 & \cdots & 0 & -y \end{vmatrix}_{k \times k} \tag{6}$$

This determinant (6) is then expanded by the elements in the first column, giving

$$D_k(y) = (-2y)A_1 - yA_2 + (-1)^{k+1} A_3 , \tag{7}$$

where A_1, A_2, and A_3 are cofactors still to be evaluated. When A_1 and A_3 are expanded by the elements in their last row and column, respectively, they become $A_1 = -yD_{k-2}(y)$ and $A_3 = (-1)^k D_{k-2}(y)$. If the determinant A_2 is expanded according to the elements of its first row, the resulting determinant according to the elements of its last row, and finally this new determinant according to the elements of its last row, one finds $A_2 = y^3 D_{k-4}(y)$. The above expressions for A_1, A_2, and A_3 are then substituted into Eq. (7), which yields the result

(8) $$D_k(y) = (2y^2 - 1)D_{k-2}(y) - y^4D_{k-4}(y) ,$$

the desired recursion formulas for the polynomials $D_k(y)$ for all $k \geq 1$.

3. THE COEFFICIENTS OF $D_n(y)$ AND RECURSION FORMULAS FOR $P(n,k)$

The polynomials $D_n(y)$ can be expanded in power series in y as

(9) $$D_n(y) = \sum_{i=0}^{n} A_{n,i}\, y^{n-i} ,$$

where $A_{n,i}$ are constants. By substituting these power series into the recursion formula (8) and equating coefficients of like powers of y, the following recursion relation among the coefficients are obtained:

(10) $$A_{n,i} = 2A_{n-2,i} - A_{n-2,i-2} - A_{n-4,i}, \quad 0 \leq i \leq n ,$$

where we take $A_{n,i} = 0$ whenever $i < 0$ or $n < i$. For $n = 2, 4, 6$ and $n = 1, 3, 5$, one obtains from the recursion formula (8),

$$D_2(y) = y^2 - y - 1$$
$$D_4(y) = y^4 - 2y^3 - 3y^2 + y + 1$$
$$D_6(y) = y^6 - 3y^5 - 6y^4 + 4y^3 + 5y^2 - y - 1$$
$$D_1(y) = -y + 1$$
$$D_3(y) = -y^3 + 2y^2 + y - 1$$
$$D_5(y) = -y^5 + 3y^4 + 3y^3 - 4y^2 - y + 1 .$$

The coefficients $A_{n,i}$ can now be evaluated by using the above polynomials and recursion relations, resulting in the set of specific formulas in addition to those of (10):

$$A_{2n,0} = 1 \qquad A_{2n+1,0} = -1$$
$$A_{2n,1} = -n \qquad A_{2n+1,1} = n + 1$$
$$A_{2n,2n} = A_{2n,2n-1} = (-1)^n \qquad A_{2n+1,2n} = -A_{2n+1,2n+1} = (-1)^{n+1}$$
$$A_{2n,2n-2} = (-1)^{n+1}(2n - 1) \qquad A_{2n+1,2n-1} = (-1)^{n+1}2n$$
$$A_{2n,2n-3} = (-1)^{n+1}(2n - 2) \qquad A_{2n+1,2n-2} = (-1)^n(2n - 1) .$$

These sets of formulas for the coefficients will then permit one to write the polynomials $D_n(y)$ as power series in y, which is very useful in obtaining recursion formulas for $P(n,k)$.

If we let $D_n(y) = 0$ in the power series expansion (9), the resulting equation implies that

$$\sum_{i=0}^{n} A_{k,i} P(n - i, k) = 0 \tag{11}$$

for all k. Then, Eq. (11) is the recursion relation for the numbers P(n,k), and for $k \leq 5$, we can write

$$P(n + 1, 1) = P(n,1)$$

$$P(n + 2, 2) = P(n + 1, 2) + P(n,2)$$

$$P(n + 3, 3) = 2P(n + 2, 3) + P(n + 1, 3) - P(n,3)$$

$$P(n + 4, 4) = 2P(n + 3, 4) + 3P(n + 2, 4) - P(n + 1, 4) - P(n,4)$$

$$P(n + 5, 5) = 3P(n + 4, 5) + 3P(n + 3, 5) - 4P(n + 2, 5) - P(n + 1, 5) + P(n,5) .$$

4. A GENERATING FUNCTION FOR THE POLYNOMIALS $D_n(y)$

Theorem 1. A generating function for $D_{2n}(y)$ is

$$[1 - y(y + 1)t][1 - (2y^2 - 1)t + y^4 t^2]^{-1} = \sum_{n=0}^{\infty} D_{2n}(y)t^n . \tag{12}$$

Proof. In Eq. (12), multiply both sides by the denominator on the left side to obtain

$$\begin{aligned} 1 - y(y + 1)t &= [1 - (2y^2 - 1)t + y^4 t^2] \cdot \sum_{n=0}^{\infty} D_{2n}(y)t^n \\ &= D_0(y) + [D_2(y) - (2y^2 - 1)D_0(y)]t \\ &\quad + \sum_{n=0}^{\infty} [D_{2n+4}(y) - (2y^2 - 1)D_{2n+2}(y) + y^4 D_{2n}(y)]\,t^{n+2} . \end{aligned} \tag{13}$$

Equating like powers of t on the left and right sides of Eq. (13), one obtains

$$\begin{aligned} D_0(y) &= 1 \\ D_2(y) &= (2y^2 - 1)D_0(y) - y(y + 1) = y^2 - y - 1 \\ D_{2n+4}(y) &= (2y^2 - 1)D_{2n+2}(y) - y^4 D_{2n}(y) \end{aligned}$$

Since these last three equations are in full agreement with the recursion formulas derived earlier, one concludes that the theorem has been proved.

Theorem 2. A generating function for $D_{2n+1}(y)$ is

$$[y + 1 - y^3t][1 - (2y^2 - 1)t + y^4t^2]^{-1} = \sum_{n=0}^{\infty} D_{2n+1}(y)t^n \quad . \tag{14}$$

The proof of this theorem readily follows if one uses the same procedure as in proving the preceding theorem.

The generating functions of Eqs. (12) and (14) can be used to obtain closed form solutions for $D_{2n}(y)$ and $D_{2n+1}(y)$ with the aid of the following equations from Rainville [2]:

$$(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \tag{15}$$

$$\begin{aligned} (a)_n &= \prod_{k=1}^{n} (a + k - 1) = a(a + 1)(a + 2) \cdots (a + n - 1), \qquad n \geq 1, \\ (a)_0 &= 1, \quad a \neq 0 \; . \end{aligned} \tag{16}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{n-k}{k} \quad . \tag{17}$$

By applying Eqs. (15) and (17) to the denominator of Eq. (12),

$$\begin{aligned} [1 - (2y^2 - 1)t + y^4t^2]^{-1} &= \sum_{n=0}^{\infty} [(1)_n/n!]\,[(2y^2 - 1)t - y^4t^2]^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2y^2 - 1)^{n-k} y^{4k} t^{n+k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y^2 - 1)^{n-2k} y^{4k} t^n \; . \end{aligned} \tag{18}$$

Hence,

$$[1 - y(y + 1)t][1 - (2y^2 - 1)t + y^4t^2]^{-1}$$

(19)
$$= \sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y - 1)^{n-2k} y^{4k} t^n + 1$$
$$- y(y + 1) \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2 - 1)^{n-2k-1} y^{4k} t^n .$$

Now

(20)
$$\sum_{n=0}^{\infty} D_{2n}(y)t^n = \sum_{n=1}^{\infty} D_{2n}(y)t^n + 1 .$$

Therefore, by equating coefficients of like powers of t in Eq. (12), a closed-form solution for $D_{2n}(y)$ is extracted:

(21)
$$D_{2n}(y) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y^2 - 1)^{n-2k} y^{4k}$$
$$- y(y + 1) \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2 - 1)^{n-2k-1} y^{4k} .$$

The closed-form solution for $D_{2n+1}(y)$ follows readily from the above derivation,

(22)
$$D_{2n+1}(y) = (y + 1) \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y^2 - 1)^{n-2k} y^{4k}$$
$$- y^3 \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2 - 1)^{n-2k-1} y^{4k} .$$

REFERENCES

1. L. Moser and M. Wyman, Problem B-6, Fibonacci Quarterly, Vol. 1, No. 1, Feb. 1963, p. 74.
2. Earl D. Rainville, Special Functions, Wiley and Sons, New York, 1960.
3. Bjarne Junge, "A Study of Matrix Developed Sequences and Polynomials," Master's Thesis, San Jose State College, San Jose, Calif., June, 1963.