THE NUMBER OF SDR'S IN CERTAIN REGULAR SYSTEMS

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ABSTRACT

Let $(a_1, \cdots, a_k) = \bar{a}$ denote a vector of numbers, and let $C(\bar{a}, n)$ denote the $n \times n$ cyclic matrix having $(a_1, \cdots, a_k, 0, \cdots, 0)$ as its first row. It is shown that the sequences $(\det C(\bar{a}, n) : n = k, k+1, \cdots)$ and $(\text{per } C(\bar{a}, n) : n = k, k+1, \cdots)$ satisfy linear homogeneous difference equations with constant coefficients. The permanent, per C, of a matrix C is defined like the determinant except that one forgets about $(-1)^{\text{sign } \pi}$ where π is a permutation.

INTRODUCTION

While she was a student at Lowell High School, Beverly Ross [2] generalized an exercise given by Marshall Hall, Jr. [1], and found an elegant solution. Hall's exercise was posed in the context of systems of distinct representatives, or SDR's for short. Let $\bar{A} = (A_1, \cdots, A_m)$ denote an m-tuple of sets, then an m-tuple $(a_1, \cdots, a_m)$ with $a_i \in A_i$ for $i = 1, \cdots, m$ is an SDR of $\bar{A}$ if the elements $a_1, \cdots, a_m$ are all distinct. Hall's exercise is the case $m = 7$ of the following problem posed and solved by Ross: Let $A_i = \{i, i+1, i+2\}$ denote a 3-set of consecutive residue classes modulo m for $i = 1, \cdots, m$. The number of SDR's of $(A_i : i = 1, \cdots, m)$ is $2 + L_m$ where L_m is the m^{th} term of the Lucas sequence 1, 3, 4, 7, 11, $\cdots$ defined by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n = 3, 4, \cdots$. For example, it follows from this result that the solution to Hall's exercise is $2 + L_7 = 31$.

In this note we give a new proof of Ross' theorem, and indicate a generalization.

ROSS' THEOREM

We shall require a simple result which appears in Ryser [3]; namely, the number of SDR's of an m-tuple $\bar{B} = (B_1, \cdots, B_m)$ of sets $B_1, \cdots, B_m$ is equal to the permanent of the incidence matrix of $\bar{B}$. Since this fact is an immediate consequence of definitions, we give them here. Let m and n denote natural numbers with $m \leq n$, and let $B_1, \cdots, B_m$ denote subsets of $\{1, \cdots, n\}$. The incidence matrix $[b(i,j)]$ of $\bar{B} = (B_1, \cdots, B_m)$ is defined by

$$b(i,j) = \begin{cases} 1, & \text{if } j \in B_i, \\ 0, & \text{if } j \notin B_i, \end{cases}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. The permanent of an $m \times n$ matrix $[r(i,j)]$ is defined to be

$$\text{per } [r(i,j)] = \sum_{\pi} r(i,\pi 1)\, r(2,\pi 2) \cdots r(m,\pi m) ,$$

where the index of summation extends over all one-to-one mappings π sending $\{1, \cdots, m\}$ into $\{1, \cdots, n\}$.

The incidence matrix C_m of the m-tuple $\overline{A} = (A_1, \cdots, A_m)$ of sets $A_1, \cdots, A_m$ considered by Ross is an $m \times m$ cyclic matrix having as its first row $(1,1,1,0,\cdots,0)$; that is, the first row has its first three components equal to 1 and the rest of its components equal to 0. For example, the incidence matrix for Hall's exercise is

$$C_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Ross' Theorem is equivalent to showing that per $C_m = 2 + L_m$. To do this, we define three sequences of matrices:

$$D_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad D_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \cdots ;$$

$$E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \cdots ;$$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \cdots .$$

Let per $C_m = c_m$, per $D_m = d_m$, per $E_m = e_m$, and per $F_m = f_m$. We use the following properties of the permanent function. First, the permanent of a 0-1 matrix is equal to the sum of the permanents of the minors of the 1's in a row or in a column of the matrix. Second, the permanent of a matrix is unchanged by permuting the rows or by permuting the columns of the matrix. Third, the permanent of a matrix having a row or column of 0's is equal to 0. Fourth, the permanent of a square matrix is equal to the permanent of the transpose of the matrix. Expanding per C_m in terms of the minors of the 1's in the first row of C_m, we find

(1) $$c_m = 2d_{m-1} + e_{m-1} \qquad (m = 4, 5, \cdots).$$

Expanding per D_m in terms of the minors of the 1's in the first column of D_m, we find

(2) $$d_m = e_{m-1} + f_{m-1} \qquad (m = 4, 5, \cdots).$$

It is easy to show that

(3) $$e_m = e_{m-1} + e_{m-2} \qquad (m = 4, 5, \cdots),$$

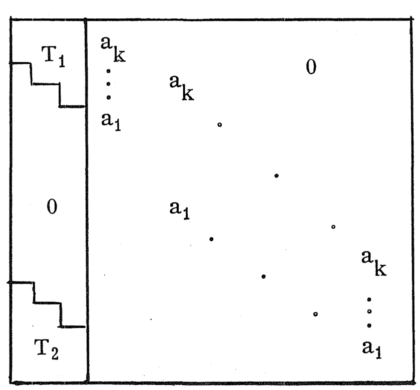
(4) $$f_m = f_{m-1} = \cdots = f_3 = 1.$$

Using the system (1)-(4) it is easy to show by induction that $e_m = F_{m+1}$, where F_m denotes the m^{th} term of the Fibonacci sequence $(1, 1, 2, 3, \cdots)$, $d_m = 1 + F_m$, and $c_m = 2 + 2F_{m-1} + F_m = 2 + F_{m-1} + F_{m+1} = 2 + L_m$ for $m = 3, 4, \cdots$.

A GENERALIZATION

Let $\bar{a} = (a_1, \cdots, a_k)$ denote a k-tuple of numbers and let T denote a $k \times (k - 1)$ matrix having all of its entries in the set $\{0, a_1, \cdots, a_k\}$. For each $n \geq k$ define an $n \times n$ matrix $C(T,n)$ as follows:

$C(T,n) =$



The first $k - 1$ columns of $C(T,n)$ have the upper triangular half T_1 of T in the upper right corner, and the lower triangular half T_2 of T in the lower left corner. All other entries in the first $k - 1$ columns of $C(T,n)$ are 0. The remaining $n - k + 1$ columns of

$C(T,n)$ consist of $n - k + 1$ cyclic shifts of the column $(a_k, \cdots, a_2, a_1, 0, \cdots, 0)$.

Given a $k \times (k-1)$ matrix T having all of its entries in $\{0, a_1, \cdots, a_k\}$ and having $(t_1, \cdots, t_{k-1})$ as its top row, we expand per $C(T,n)$ by the minors of elements in the top row of $C(T,n)$. It turns out that these minors always have the form $C(T_i, n-1)$ where T_i is a $k \times (k-1)$ matrix having all its entries in $\{0, a_1, \cdots, a_k\}$. Thus, there exist $k \times (k-1)$ matrices $T, \cdots, T$ having all their entries in $\{0, a_1, \cdots, a_k\}$ such that

$$\text{per } C(T,n) = \sum_{i=1}^{k} t_i \text{ per } C(T_i, n-1) , \tag{1}$$

where $t_k = a_k$. (If we are dealing with determinants, $(-1)^i$ must be put into the summand.)

We have an equation like (1) for each matrix T; hence, we have a finite system of equations if we let T range over all possible $k \times (k-1)$ matrices with their entries in $\{0, a_1, \cdots, a_k\}$. The existence of this system of difference equations implies the existence of a difference equation satisfied by the sequence (per $C(T,n) : n = k, k+1, \cdots$) for each fixed matrix T. (This is also true for the sequence ($\det C(T,n) : n = k, k+1, \cdots$).) A consequence of the foregoing is the result proved by Ross, but evidently much more is true.

Let $r_1, \cdots, r_n$ denote natural numbers with $1 = r_1 < \cdots < r_n = k$, and for each natural number $m \geq k$ define the collection $\overline{A}_m = \{A_1, \cdots, A_m\}$ of sets A_i of residue classes modulo m where

$$A_i = \{r_1 + i, \cdots, r_n + i\} .$$

Let $a(m)$ denote the number of SDR's of $\overline{A}_m$, then the sequence $(a(m) : m = k, k+1, \cdots)$ satisfies a linear homogeneous difference equation with constant coefficients. The proof of this fact follows the proof of Ross' Theorem given in the preceding section.

Note that our existence theorem has a constructive proof, but we do not have an explicit expression for a difference equation satisfied by the sequence (per $C(T,n) : n = k, k+1, \cdots$). This gives rise to a host of interesting research problems. For example, give a difference equation satisfied by the sequence (per $C(k,n) : n = k, k+1, \cdots$) where $C(k,n)$ is the cyclic $n \times n$ matrix having as its first row $(1, \cdots, 1, 0, \cdots, 0)$ consisting of k 1's followed by $n - k$ 0's.

REFERENCES

1. Marshall Hall, Jr., Combinatorial Theory, Blaisdell Publishing Company, Waltham, Mass., 1967. (Problem 1, page 53.)
2. Beverly Ross, "A Lucas Number Counting Problem," Fibonacci Quarterly, Vol. 10 (1972), pages 325-328.
3. Herbert J. Ryser, "Combinatorial Mathematics," Number 14 of the Carus Mathematical Monographs, John Wiley, 1963.