REMARK ON A PAPER BY DUNCAN AND BROWN ON THE SEQUENCE OF LOGARITHMS OF CERTAIN RECURSIVE SEQUENCES

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In the present paper, it is shown that the main theorem in [1], see p. 484, can be established by using one of J. G. van der Corput's difference theorems [2]. Moreover, by using a theorem of C. L VandenEynden [3] we show the property that the sequence of the integral parts of the logarithms of the recursive sequence under consideration is also uniformly distributed modulo m for any integer $m \geq 2$.

Lemma 1. Let (x_n), $n = 1, 2, \cdots$, be a sequence of real numbers. If

$$\lim_{n \to \infty} (x_{n+1} - x_n) = \alpha ,$$

α irrational, then (x_n) is u.d. mod 1 ([2], p. 378).

Lemma 2. Let (x_n), $n = 1, 2, \cdots$, be a sequence of real numbers. Assume that the sequence (x_n/m), $n = 1, 2, \cdots$, is u.d. mod 1 for all integers $m \geq 2$. Then the sequence of the integral parts $([x_n])$, $n = 1, 2, \cdots$, is u.d. mod m [3].

For the notion of uniform distribution modulo m we refer to [4].

Theorem. Let (V_n), $n = 1, 2, \cdots$, be a sequence generated by the recursion relation

$$V_{n+k} = a_{k-1}V_{n+k-1} + \cdots + a_1V_{n+1} + a_0V_n, \quad n \geq 1 , \tag{1}$$

where $a_0, a_1, \cdots, a_{k-1}$ are non-negative rational coefficients with $a_0 \neq 0$, k is a fixed integer, and

$$V_1 = \gamma_1, \quad V_2 = \gamma_2, \quad \cdots, \quad V_k = \gamma_k \tag{2}$$

are given positive values for the initial terms. It is assumed that the polynomial

$$x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$$

has k distinct real roots $\beta_1, \beta_2, \cdots, \beta_k$ satisfying $0 < |\beta_k| < \cdots < |\beta_k|$ and such that none of the roots has magnitude equal to 1. Then:

1. The sequence $(\log V_n)$, $n = 1, 2, \cdots$, is u.d. mod 1 [1].
2. The sequence $([\log V_n])$, $n = 1, 2, \cdots$, is u.d.

Proof. By (1) and (2), we have that

$$V_n = \sum_{j=1}^{k} \alpha_j \beta_j^n \quad (n \geq 1),$$

where the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_k$ are uniquely determined by assumption (2). Let p be the largest value of j for which $a_j \neq 0$. We have $p \geq 1$. Hence

$$V_n = \sum_{j=1}^{p} \alpha_j \beta_j^p .$$

Now

$$\frac{V_{n+1}}{V_n} = \frac{\alpha_1 \beta_1^{n+1} + \cdots + \alpha_p \beta_p^{n+1}}{\alpha_1 \beta_1^n + \cdots + \alpha_p \beta_p^n} \to \beta_p \quad \text{as} \quad n \to \infty ,$$

since $\beta_1^n \mid \beta_p^n \to 0$ (as $n \to \infty$), $i = 1, 2, \cdots, p-1$, because of the conditions on the absolute values of the β_j. (From the conditions follows that $\beta_p > 0$.) Hence we have that

$$\log V_{n+1} - \log V_n \to \log \beta_p, \quad \text{as} \quad n \to \infty .$$

The number β_p is algebraic and therefore $\log \beta_p$ is an irrational number (see [1]). Hence Lemma 1 applies and we obtain that the sequence $(\log V_n)$ is u.d. mod 1. This proves Duncan and Brown's result.

In order to show the second part of the theorem we observe that for every integer $m \geq 2$

$$\frac{\log V_{n+1}}{m} - \frac{\log V_n}{m} \to \frac{\beta_p}{m}, \quad \text{as} \quad n \to \infty ,$$

hence the sequence $((\log V_n)/m)$, $n = 1, 2, \cdots$ is u.d. mod 1, and according to Lemma 2 we obtain that the sequence of the integral parts $([\log V_n])$ is u.d. mod m for every integer $m \geq 2$.

Remark. By restricting the order of the recurrence we may relax the conditions on the coefficients a_j and the initial values of V_n. The values of elements of (V_n) can be negative in that case, and so we obtain a result regarding the logarithms of the absolute value of V_n.

Let (V_n), $n = 1, 2, \cdots,$ be a sequence generated by the recurrence

$$V_{n+2} = a_1 V_{n+1} + a_0 V_n, \quad n \geq 1,$$

where $V_1 = \gamma_1$, $V_2 = \gamma_2$. We assume that γ_1, γ_2, a_0 and a_1 are rational numbers, where γ_1 and γ_2 are $\neq 0$, and a_0 and a_1 not both 0. Moreover, it is assumed that the polynomial $x^2 - a_1 x - a_0$ has distinct real roots, β_1 and β_2, one of which has an absolute value different from 1. Then the sequence $(\log|V_n|)$ is u.d. mod 1, and the sequence of integral parts $([\log|V_n|])$ is u.d.

Proof. We have

$$V_n = \frac{(\gamma_2 - \gamma_1\beta_2)\beta_1^{n-1} - (\gamma_2 - \gamma_1\beta_1)\beta_2^{n-1}}{\beta_1 - \beta_2} ,$$

where

$$\beta_1 = \tfrac{1}{2}(a_1 + \sqrt{a_1^2 + 4a_0}), \qquad \beta_2 = \tfrac{1}{2}(a_1 - \sqrt{a_1^2 + 4a_0}) .$$

Now

$$\log|V_{n+1}| - \log|V_n| = \log\left|\frac{(\gamma_2 - \gamma_1\beta_2)\beta_1^{n} - (\gamma_2 - \gamma_1\beta_1)\beta_2^{n}}{(\gamma_2 - \gamma_1\beta_2)\beta_1^{n-1} - (\gamma_2 - \gamma_1\beta_1)\beta_2^{n-1}}\right| .$$

We may suppose that $|\beta_1| \neq 1$, $|\beta_2/\beta_1| < 1$.

Since $\log|V_{n+1}| - \log|V_n| \to \log|\beta_1|$ as $n \to \infty$, and as $|\beta_1|$ is algebraic when β_1 is algebraic, we may complete the proof in the same way as done above.

REFERENCES

1. J. L Brown and R. L. Duncan, "Modulo One Uniform Distribution of the Sequence of Logarithms of Certain Recursive Sequences," Fibonacci Quarterly, Vol. 8, No. 5 (1970), pp. 482, etc.
2. J. G. van der Corput, "Diophantische Ungleichungen," Acta. Mathematica, Bd. 56 (1931), pp. 373-456.
3. C. L. VandenEynden, The Uniform Distribution of Sequences, Ph.D. Thesis, University of Oregon, 1962.
4. I. Niven, "Uniform Distribution of Sequences of Integers," Trans. A.M.S.,

ERRATA

Please make the following changes in the article, "A Triangle with Integral Sides and Area," by H. W. Gould, appearing in Vol. 11, No. 1, pp. 27-39.

Page 28, line 3 from bottom:	For	$+ u - v\sqrt{3})$	read	$+ (u - v\sqrt{3})$.
Page 31, Eq. (11):	For	$\frac{K^2}{a^2}$	read	$\frac{K^2}{s^2}$.
Page 31, line 6 from bottom:	For	$4x^2 - 3y^2$	read	$4x^2 - 3v^2$.
Page 33, Eq. (17):	For	r_u^2	read	r_a^2 .
Page 33, Eq. (22):	For	r_c: , 6, 14	read	$r_c{:}\infty$, 6, 14 .
Page 35, Line 13:	For	i.e.	read	as .
Page 35, Line 16:	For	N = orthocenter	read	H = orthocenter.
Page 35, line 9 from bottom:	For	$\|I = H\|^2$	read	$\|I - H\|^2$.
Page 36, line 12 from bottom:	For	residue	read	radius .

Page 39, Ref. 4. Underline Jahrbuch uber die.

Page 39, Ref. 4. Closed quotes should follow sind rather than Dreieck.