# A PROCEDURE FOR THE ENUMERATION OF 4 x n LATIN RECTANGLES

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Let $M_n^r$ denote the number of normalized (first row in natural order) $r \times n$ Latin rectangles, $M_n^1 = 1$ for all n. The $M_n^2$ are the rencontre numbers [1]. Several methods are available for computing the $M_n^3$ [1, 2, 3]. In this report, a procedure is presented that is effective in finding $M_n^r$ for $r \leq 4$.

## CALCULATION OF $M_n^2$ AND GENERAL FORMULA FOR $M_n^r$

Consider the diagram (Fig. 1, drawn, as are all the diagrams, for n = 5) consisting of an $n \times n$ square, made up of $n^2$ cells arranged in n rows and n columns. Label each cell with the numerical indices giving its position in the square, writing them as a one-column matrix with the row index at the top, and mark those cells whose 2 indices are different. Call the marked cells good, the others bad. (For any r, the cells called good will be those with all indices different.) Now "expand" the diagram into "terms" by taking one cell from each row and each column, keeping the row indices in natural order. Each term then corresponds uniquely to a $2 \times n$ rectangular array whose first row is in natural order; a term made up entirely of good cells corresponds to a normalized Latin rectangle. $M_n^2$ is the number of such "all-good" terms in the expansion. Using the principle of inclusion and exclusion, one can read off $M_n^2$ from the diagram at sight (cf [4]). The expression so obtained is formula (1), below, with $A_{r,n}^k = n^{(k)}$, where $n^{(k)} \equiv n(n-1)(n-2) \cdots (n-k+1)$.

For $r > 2$, one can prepare an analogous diagram of an r-dimensional hypercube (e.g., Figs. 2 and 4), referred to in this report as the $n^r$-cube, and obtain the formula, valid for all $r > 0$.

$$(1) \qquad M_n^r = \sum_{k=0}^{n} (-1)^k \frac{A_{r,n}^k}{k!} \left[(n-k)!\right]^{r-1} .$$

Here $A_{r,n}^k$ denotes the number of k-tuples in the expansion of the $n^r$-cube. (The word k-tuple, in this report, will always mean an ordered set of k bad cells, no two of which are from the same dimensional level.) $A_{r,n}^0 = 1$ for all r and all n, by definition.

## CALCULATION OF $M_n^3$

The diagram is shown in Fig. 2. The layers are numbered from left to right and the indices are written in the order: layer, row, column. (Column matrices of indices will be written in the form $(a, b, c)'$ for typographical convenience.) There are two types of bad cells: $\alpha'$-cells, having all 3 indices alike, and $\beta'$-cells, having exactly 2 indices alike. The

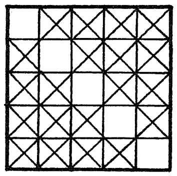

Fig. 1 $5^2$-Cube. (Good cells outlined and cross-hatched)

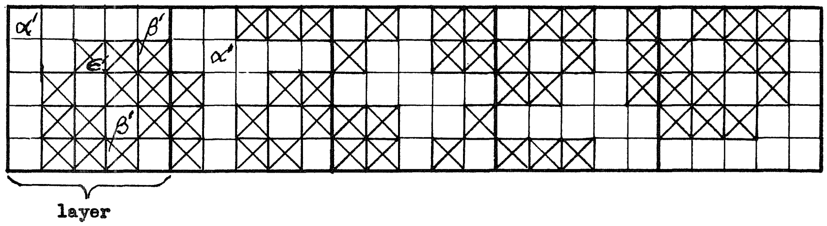


Fig. 2 $5^3$-Cube (which is also the $5^3 - \alpha'$-Cube). (Good cells outlined and cross-hatched.)

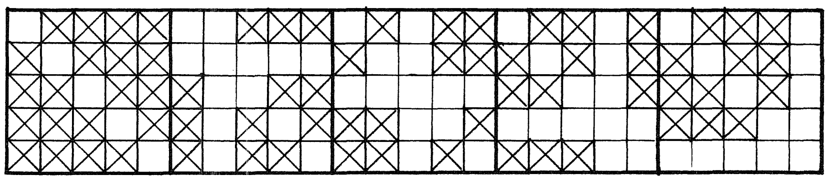

Fig. 3 $5^3 - \beta'$-Cube After $(2,1,1)'$. (Good cells outlined and cross-hatched.)

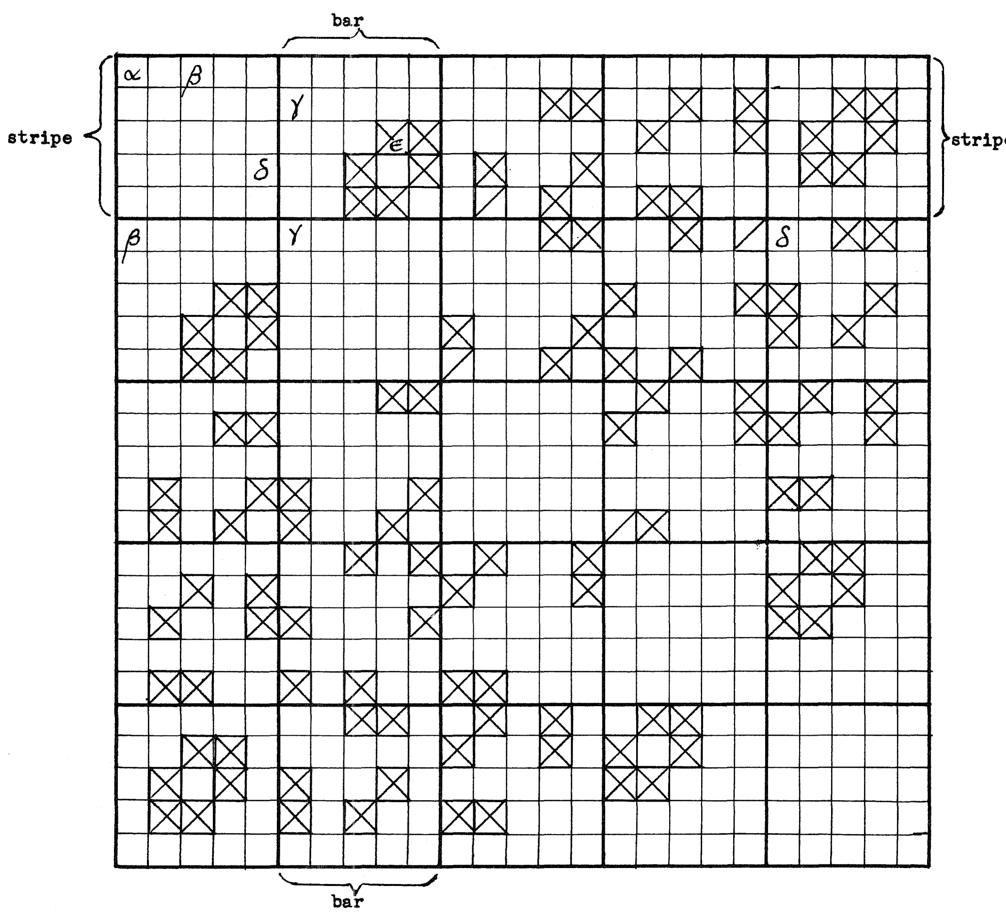


Fig. 4 $5^4$-Cube (which is also the $5^4 - \alpha$-Cube. (Good cells outlined and cross-hatched.)

Table 1

Formulas for the $\omega_n^k$

(Formulas are valid for $k > 0$ and for $n$ as indicated. See text for $k = 0$. Except for $\alpha_1^1$, all $\omega_n^k$ are 0 for $k > 0$ and for values of $n > 0$ not covered by the formulas.)

$$\alpha_n^k = n\left\{\alpha_{n-1}^{k-1} + (n-1)\left[4\beta_{n-1}^{k-1} + 3\gamma_{n-1}^{k-1} + 6(n-2)\delta_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$(\alpha_1^1 = 1)$$

$$\beta_n^k = \alpha_n^k - 3k(n-1)(n-2)\delta_{n-1}^{k-1} \qquad (n \geq 1)$$

$$\gamma_n^k = \beta_n^k - k(n-1)(n-2)(2\eta_{n-1}^{k-1} - \delta_{n-1}^{k-1}) \qquad (n \geq 1)$$

$$\delta_n^k = \beta_n^k - k(n-2)\left[2(\theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1} - \delta_{n-1}^{k-1}) + (n-3)(\mu_{n-1}^{k-1} + 2\xi_{n-1}^{k-1} - \epsilon_{n-1}^{k-1})\right] \qquad (n \geq 2)$$

$$(n+1)^{(4)}\epsilon_n^k = (n-k+1)^4\alpha_{n+1}^k - \alpha_{n+1}^{k+1} \qquad (n \geq 3)$$

$$\zeta_n^k = \gamma_n^k - k(n-1)\left[2\beta_{n-1}^{k-1} + (n-2)\delta_{n-1}^{k-1}\right] \qquad (n \geq 1)$$

$$\eta_n^k = \delta_n^k - k\left\{2\gamma_{n-1}^{k-1} + (n-2)\left[4\delta_{n-1}^{k-1} + (n-3)\epsilon_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$\theta_n^k = \delta_n^k - k\left\{2\zeta_{n-1}^{k-1} + (n-2)\left[2\eta_{n-1}^{k-1} + \theta_{n-1}^{k-1} + 2\lambda_{n-1}^{k-1} + \nu_{n-1}^{k-1} + (n-3)(\mu_{n-1}^{k-1} + \xi_{n-1}^{k-1})\right]\right\} \qquad (n \geq 2)$$

$$\lambda_n^k = \delta_n^k - 2k\left\{\zeta_{n-1}^{k-1} + (n-2)\left[\eta_{n-1}^{k-1} + \theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1} + (n-3)\mu_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$\mu_n^k = \epsilon_n^k - 2k\left\{2(\theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1}) + (n-3)\left[3\mu_{n-1}^{k-1} + 2\xi_{n-1}^{k-1} + (n-4)\pi_{n-1}^{k-1}\right]\right\} \qquad (n \geq 3)$$

$$\nu_n^k = \beta_n^k - 3k(n-2)\left[2\theta_{n-1}^{k-1} + (n-3)\mu_{n-1}^{k-1}\right] \qquad (n \geq 2)$$

$$3(n-1)(n-2)\xi_n^k = \alpha_n^k + 2(2n-1)\beta_n^k + 3n\gamma_n^k + (6n-3k)(n-1)\delta_n^k - \left\{6\zeta_n^k + (n-1)\left[6(\eta_n^k + \theta_n^k + \lambda_n^k) + 3(n-2)\mu_n^k + \nu_n^k\right]\right\} \qquad (n \geq 3)$$

$$n^{(4)}\pi_n^k = (n-k+1)^4\beta_{n+1}^k - \beta_{n+1}^{k+1} - n(n-1)\left[3\delta_n^k + (n-2)(\epsilon_n^k + 3\mu_n^k)\right] \qquad (n \geq 4)$$

good cells (not needed in the calculations when $r = 3$) are $\epsilon'$-cells. The numbers of each cell-type in any layer are easily ascertained.

Letting $\omega'$ stand for either of $\alpha'$, $\beta'$, denote as an $n^3 - \omega'$-cube the cube that is obtained when all cells in the layer, row and column occupied by a chosen $\omega'$-cell of the $(n + 1)^3$-cube are removed and the remaining parts of the diagram are allowed to collapse upon themselves (i.e., "ranks are closed"), the good or bad nature of each cell being preserved. Let $\omega'^k_n$ denote the number of k-tuples in an $n^3 - \omega'$-cube. It is apparent that the $n^3 - \alpha'$-cube is identical with the $n^3$-cube and that, for $k > 1$,

$$A^k_{3,n} \equiv \alpha'^k_n = n\alpha'^{k-1}_{n-1} + 3n(n - 1)\beta'^{k-1}_{n-1} \, . \tag{2}$$

($\alpha'^1_n = n^3 - n^{(3)}$ and $\beta'^1_n = \alpha'^1_n - 2(n - 1)$, by direct count in the diagram.)

To find $\beta'^k_n$, we can clearly use any $\beta'$-cell in the $(n + 1)^3$-cube. Choosing $(2,1,1)'$, we get the $n^3 - \beta'$-cube of Fig. 3. It differs from the $n^3 - \alpha'$-cube only in the first layer, all the differing cells being $\beta'$-cells in the $n^3 - \alpha'$-cubes. Accordingly,

$$\beta'^k_n = \alpha'^k_n - 2k(n - 1)\beta'^{k-1}_{n-1} \, . \tag{3}$$

The factor k in the second term on the right enters because of the way in which "k-tuple" has been defined.

All $A^k_{3,n}$ can now be calculated, for any given n; when they are substituted in (1), $M^3_n$ is obtained.

## CALCULATION OF $M^4_n$

The diagram for the $n^4$-cube is shown in Fig. 4. The dimensional levels are to be written in the order: stripe, bar, row, column; the intersection of stripes a and b is called the field $[a, b]$. The procedure is analogous to that used above for the case $r = 3$. There are now, besides the good or $\epsilon$-cells, four kinds of bad cells: $\alpha$-cells, with all indices alike; $\beta$-cells, with exactly three indices alike; $\gamma$-cells, with two distinct pairs of like indices, and $\delta$-cells, with exactly two indices alike.

The numbers of cells of each type are again easily found, and we have at once the formula for $A^k_{4,n} \equiv \alpha^k_n$ given in Table 1. $\beta^k_n$ is obtained from the $n^4$-$\beta$-cube resulting from choosing $(2,1,1,1)'$ in the $(n + 1)^4$-cube (second formula in Table 1), just as $\beta'^k_n$ was obtained in the case $r = 3$. $\gamma^k_n$ and $\delta^k_n$ are not so immediate, but can be found by the method outlined below.

We first analyze the $n^4 - \beta$-cube much as we did the $n^4 - \alpha$-cube. That is to say, we study the types and topographical distributions of the second members of those pairs of cells of the $(n + 1)^4$-cube whose first member is a selected $\beta$-cell. The $\beta$-cell chosen in the $(n + 1)^4$-cube to obtain the results shown in Fig. 5 is $(2,1,1,1)'$. It is useful to observe that, in a pair of columns of indices, the two numbers in a row may be interchanged without affecting the properties in which we are interested. There prove to be 13 different cell-types, including the five already observed. They are designated by Greek letters, as shown in Fig. 5. (The cells in the first stripe all retain the same designations they had in the $n^4 - \alpha$-cube;

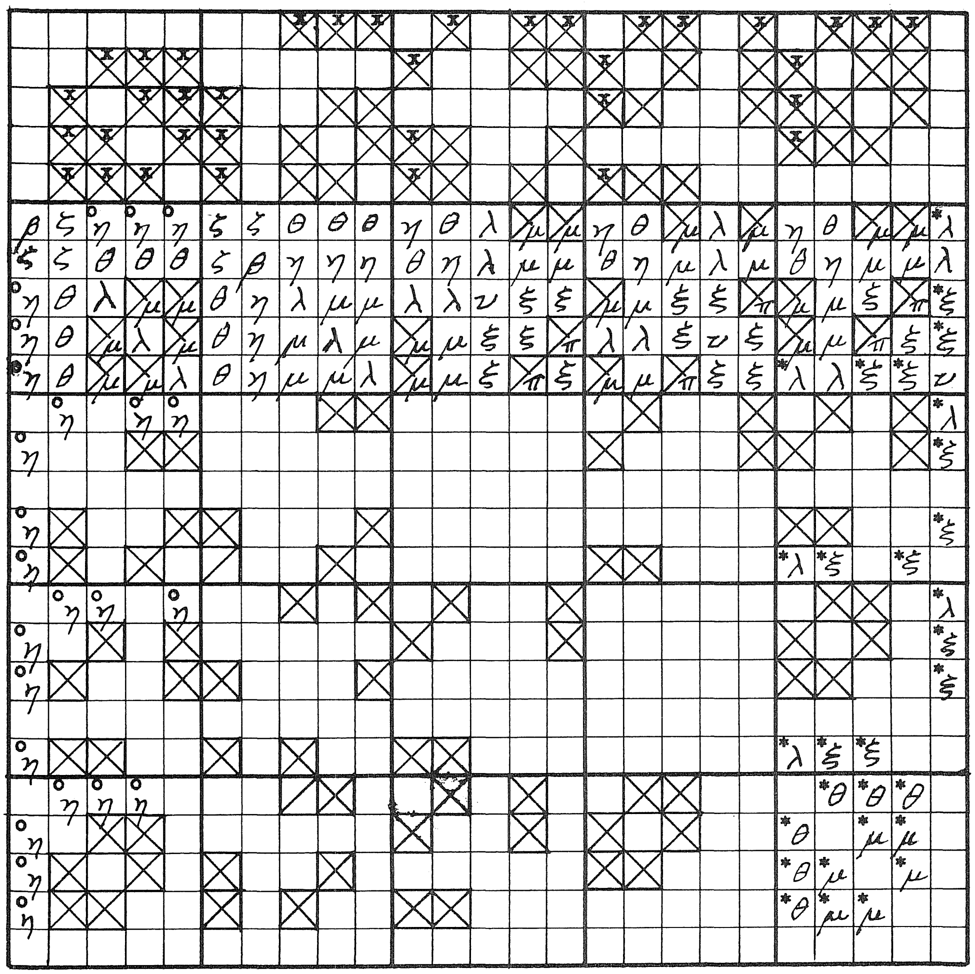

Fig. 5 $5^4$ - $\beta$-Cube after $(2,1,1,1)'$. (Good cells outlined and cross-hatched.) x indicates new good cell; o indicates where good cell is to be inserted to get $\gamma_5^k$. * indicates where good cell is to be inserted to get $\delta_5^k$ (see text).

each of the lower stripes has the same numbers of cells of types $\beta, \zeta, \cdots, \pi$ as the second stripe, but distributed differently.)

Now, to find $\gamma_n^k$, note that the $n^4$ - $\gamma$-cube that results from choosing $(2,2,1,1)'$ can be obtained from the $n^4$ - $\beta$-cube (Fig. 5) by removing all the good cells from the field $[1,1]$ and inserting good cells at those places in the first bar indicated with o in the diagram. This leads to the third formula in Table 1.

To get $\delta_n^k$, choose $(2, n+1, 1\ 1)'$ and, in the resulting $n^4$ - $\delta$-cube jump the first bar over all the others, so that it becomes the $n^{th}$ bar (this will not affect the expansion of the cube). This adjusted $n^4$ - $\delta$-cube may be obtained from the $n^4$ - $\beta$-cube of Fig. 5 by removing all the good cells in $[1,n]$ and inserting good cells at the places in the $n^{th}$ bar indicated with *. Thus the fourth formula of Table 1 is obtained.

$\epsilon_n^k$ is found by considering all (rather than only the bad) cells of the $n^4$-cube, and proceeding as in the derivation of $\alpha_n^k$.

Formulas for $\zeta_n^k, \cdots, \pi_n^k$ remain to be derived, in order to make the results for $\gamma_n^k$ and $\delta_n^k$ effective. All but $\xi_n^k$ and $\pi_n^k$ may be found in the following way, $\lambda_n^k$ being used as example. Choose an appropriate $\lambda$-cell, such as $(2,1,3,3)'$ in the $(n+1)^4$ - $\beta$-cube, noting that the chosen cell is a $\delta$-cell in the $(n+1)^4$ - $\alpha$-cube. Now adjust $\delta_n^k$ by correcting for the "new" good cells, marked x in Fig. 5 (i.e., cells that are good in the $\beta$-cube but bad in the $\alpha$-cube). The cell pairs that must be examined in this process all prove to be reducible to pairs of the kinds already introduced. There results, finally, the formula for $\lambda_n^k$ in Table 1. All the rest of the formulas of Table 1 except the last two are derived in the same way.

If the choice of the $\lambda$-cell had been inappropriate (e.g., $(2,2,3,3)'$), the procedure would have met an impasse and have failed. This happens for all choices of $\xi$-cells and $\pi$-cells, $\xi_n^k$ can be found, however, by equating the result already known for $\beta_{n+1}^{k+1}$ with that obtained by expanding the $(n+1)^4$ - $\beta$-cube in terms of the $\alpha_n^k, \cdots, \zeta_n^k, \cdots, \xi_n^k, \pi_n^k$. The latter expansion is analogous to that used to find $\alpha_n^k$, above. $\pi_n^k$ is found by using all, rather than only the bad, cells in the diagram, by analogy with the derivation of $\epsilon_n^k$. There result the last two formulas of Table 1.

As an initial set of values for the recurrences of Table 1, one can use the $\omega_n^0$, whose values are: for $n > 3$, all $\omega_n^0$ are 1; for $n = 3$, $\pi_3^0$ is 0 and all others are 1; for $n = 2$, $\epsilon_2^0, \mu_2^0, \xi_2^0$ and $\pi_2^0$ are 0 and all others are 1; for $n = 1$, $\alpha_1^0$, $\beta_1^0$, $\gamma_1^0$ and $\delta_1^0$ are 1 and all the rest are 0. The $\omega_n^1$ can be checked by direct count in the appropriate diagram.

For any given $n > 0$, all the $A_{4,n}^k$ can now be calculated, and $M_n^4$ can be found by substituting them in (1). Some enumerations of $4 \times n$ Latin rectangles obtained by the method here presented are:

$$M_4^4 = 24$$
$$M_5^5 = 1,344$$
$$M_6^4 = 393,120$$
$$M_7^4 = 155,185,920$$
$$M_8^4 = 143,432,634,240 \ .$$

## REFERENCES

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