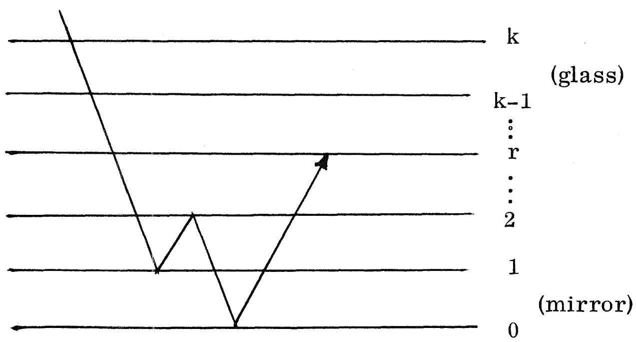
MULTIPLE REFLECTIONS

LEO MOSER and MAX WYMAN
University of Alberta, Edmonton, Alberta, Canada

Let us consider a set of k parallel plane prisms and a mirror arranged as follows:



Let $f_r(n)$ be the number of paths which start at the top of the plate and reach plate r after n upward reflections. Further, let $A = (a_{ij})$ be a $k \times k$ enumerating matrix such that

(1.1) $$f_r(n) = \sum_{j=1}^{k} a_{rj} f_j(n-1) .$$

If F(n) denotes the one column matrix

(1.2) $$F(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \\ \cdots \\ f_k(n) \end{bmatrix}$$

then (1.1) can be written

(1.3) $$F(n) = AF(n-1) .$$

Hence, by iteration we have

(1.4) $$F(n) = A^n F(0) .$$

Thus (1.4) provides an explicit solution for F(n) in terms of F(0). This form is not suitable to compute the asymptotic behavior of F(n) for large values of n. We now derive a second explicit form by means of which the asymptotic behavior is easily calculated.

The characteristic equation of the matrix A is

$$|\lambda I - A| = \lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_0 = 0 . \tag{1.5}$$

Let us assume that the roots of (1.5) are $\lambda_1, \lambda_2, \cdots, \lambda_k$ and that these roots are distinct. The case of multiple roots can also be treated easily by the method we shall use.

Since every matrix A satisfies its own characteristic equation we also have

$$A^k + c_{k-1}A^{k-1} + \cdots + c_0 I = 0 . \tag{1.6}$$

Multiplying by A^{n-k}, we have

$$A^n + c_{k-1}A^{n-1} + \cdots + c_0 A^{n-k} = 0. \tag{1.7}$$

From (1.4) and (1.7) we immediately have

$$F(n) + c_{k-1}F(n-1) + \cdots + c_0 F(n-k) = 0 , \tag{1.8}$$

and

$$f_r(n) + c_{k-1}f_r(n-1) + \cdots + c_0 f_r(n-k) = 0 . \tag{1.9}$$

However, Eq. (1.9) depends for its solution on the equation

$$\lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_0 = 0 . \tag{1.10}$$

Hence, the general solution for (1.9) is

$$f_r(n) = \sum_{j=1}^{k} B_{rj}\lambda_j^n , \tag{1.11}$$

where the constants B_{rj} do not depend on n. Since k is considered fixed we may consider the matrices 1, A, $\cdots$, A^{k-1} as having been computed. Hence from (1.4) and the boundary conditions we may consider $f_r(0)$, $f_r(1)$, $\cdots$, $f_r(k-1)$ as being known. Hence from (1.11) we will have k equations that determine $B_{r1}, B_{r2}, \cdots, B_{rk}$. Explicit expressions for these constants can be given. From (1.10) we can easily see the asymptotic behavior of $f_r(n)$. Let us write λ_j in the form $\lambda_r = r_j \exp(i\theta_j)$. Further let us assume $r_1 = r_2 = \cdots = r_p > r_{p+1} > r_{p+2} > \cdots > r_k$. Clearly,

$$f_r(n) \sim r_1^n \sum_{j=1}^{k} B_{ij} \exp(in\theta_j) . \tag{1.12}$$

If p = 1 then

$$f_r(n) \sim \lambda_1^n B_{r1} . \tag{1.13}$$

Example. If the matrix A is given by

(1.14)
$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & k \end{bmatrix}$$

the characteristic determinant

(1.15)
$$D_k(\lambda) = |\lambda I - A|$$

can be shown to satisfy the recurrence relation

(1.16)
$$D_k(\lambda) = (1 - 2\lambda)D_{k-1}(\lambda) - \lambda^2 D_{k-2}(\lambda), \quad D_0(\lambda) = 1, \quad D_1(\lambda) = 1 - \lambda .$$

The solution of (1.16) is easily obtained to be

(1.17)
$$\begin{aligned} D_k(\lambda) &= H\left(\frac{(1 - 2\lambda) + \sqrt{(1 - 2\lambda)^2 - 4\lambda^2}}{2}\right)^k \\ &\qquad + K\left(\frac{(1 - 2\lambda) - \sqrt{(1 - 2\lambda)^2 - 4\lambda^2}}{2}\right)^k \\ &= HR_1 + KR_2 \qquad , \end{aligned}$$

where H, K are constants depending on λ but not on k. Filling the boundary conditions, we find

(1.18)
$$D_k(\lambda) = (R_1 + R_2)/2 + (R_1 - R_2)/(2\sqrt{(1 - 2\lambda)^2 - 4\lambda^2}) ,$$

where R_1 and R_2 are defined in (1.17).

In order to find the roots of $D_k(\lambda) = 0$ we make the substitution

(1.19)
$$\lambda = \frac{1}{2(1 + \cos\theta)} = \frac{1}{4}\sec^2\frac{\theta}{2} .$$

Hence (1.19) becomes

$$\cos k\theta + \frac{1 + \cos\theta}{\sin\theta}\sin k\theta = 0 ,$$

$$\frac{\sin(k + \frac{1}{2})\theta}{\sin\theta}\cos\tfrac{1}{2}\theta = 0 .$$

Obviously, the roots of (1.19) are

(1.20)
$$\theta = \frac{s\pi}{k + \frac{1}{2}}, \qquad s = 1, 2, \cdots ,$$

where $s = 0$ must be excluded because of the denominator. Hence the roots of $D_k(\lambda) = 0$ are given by

(1.21) $$\lambda = \tfrac{1}{4}\sec^2\left(\frac{s\pi}{2k+1}\right), \qquad s = 1, 2, \cdots .$$

Obviously only k are distinct, and arranged in order of magnitude we have

(1.22) $$\lambda_1 = \tfrac{1}{4}\sec^2\left(\frac{k\pi}{2k+1}\right), \quad \lambda_2 = \tfrac{1}{4}\sec^2\left(\frac{(k-1)\pi}{2k+1}\right), \cdots, \lambda_k = \tfrac{1}{4}\sec^2\left(\frac{\pi}{2k+1}\right).$$

Thus

(1.23) $$f_r(n) \sim \left(\tfrac{1}{2}\sec\left(\frac{\pi}{2k+1}\right)\right)^n B_{r1} .$$

TWO NUMERICAL CASES

Case 1. $k = 2$.

$$D_2(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1 = 0 ,$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{3-\sqrt{5}}{2},$$

$$f_1(0) = 0, \quad f_2(0) = 1, \quad f_1(1) = 1, \quad f_2(1) = 2 ,$$

$$f_1(n) = B_{11}\lambda_1^n + B_{12}\lambda_2^n ,$$

$$\begin{cases} 0 = B_{11} + B_{12} \\ 1 = B_{11}\lambda_1 + B_{12}\lambda_2 \end{cases}$$

$$\therefore \quad B_{11} = \frac{1}{\lambda_1 - \lambda_2} \quad \text{and} \quad B_{12} = \frac{1}{\lambda_1 - \lambda_2} .$$

$$\therefore \quad f_1(n) = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\lambda_1^n(1 - \lambda_2^{2n}) \sim \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n$$

Similarly,

$$f_2(n) = B_{21}\lambda_1^n + B_{22}\lambda_2^n$$

$$1 = B_{21} + B_{22}$$

$$2 = B_{21}\lambda_1 + B_{22}\lambda_2$$

$$\therefore \quad B_{21} = \frac{2-\lambda_2}{\lambda_1-\lambda_2} = \frac{1+\sqrt{5}}{2\sqrt{5}}, \qquad B_{22} = \frac{2-\lambda_1}{\lambda_2-\lambda_1} = -\frac{1-\sqrt{5}}{2\sqrt{5}} .$$

$$\therefore f_2(n) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}[\alpha^{2n+1} - \beta^{2n+1}] = F_{2n+1}$$

which gives the complete solution.

Case 2. k = 3.

$$\lambda_1 = \tfrac{1}{4}\sec^2\left(\frac{3\pi}{7}\right), \qquad \lambda_2 = \tfrac{1}{4}\sec^2\left(\frac{2\pi}{7}\right), \qquad \lambda_3 = \tfrac{1}{4}\sec^2\left(\frac{\pi}{7}\right)$$

$$\begin{array}{lll} f_1(0) = 0 & f_2(0) = 0 & f_3(0) = 1 \\ f_1(1) = 1 & f_2(1) = 2 & f_3(1) = 3 \\ f_1(2) = 6 & f_2(2) = 11 & f_3(2) = 14 \end{array} .$$

Thus

$$\begin{aligned} f_3(n) &= B_{31}\lambda_1^n + B_{32}\lambda_2^n + B_{33}\lambda_3^n \\ 1 &= B_{31} + B_{32} + B_{33} \\ 3 &= B_{31}\lambda_1 + B_{32}\lambda_2 + B_{33}\lambda_3 \\ 14 &= B_{31}\lambda_1^2 + B_{32}\lambda_2^2 + B_{33}\lambda_3^2 \end{aligned} .$$

Solving simultaneously,

$$B_{31} = \frac{\lambda_2\lambda_3 - 3(\lambda_2 + \lambda_3) + 14}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} .$$

Calculating $\lambda_1, \lambda_2, \lambda_3$ and substituting above gives $B_{31} \doteq 0.537$, so that

$$f_3(n) \sim 0.537\left(\tfrac{1}{2}\sec\left(\frac{3\pi}{7}\right)\right)^{2n} .$$

[Continued from page 301.]

Page 49, Eq. (33): Please change the last number on the line from "3" to "1."

Page 49, Line following Eq. (34): Please raise "(mod 3)" to the main line of type.

Page 49, line 6 from bottom: Please insert brackets around X(X - 1), X.

Page 53, line 2 from bottom: In the third column from the left, please change the number to read: " 2 750 837 603 ."