NOTES ON BINOMIAL COEFFICIENTS: IV - PROOF OF A CONJECTURE OF GOULD ON THE GCD'S OF TWO TRIPLES OF BINOMIAL COEFFICIENTS

DAVID SINGMASTER
Polytechnic of the South Bank, London, and
Instituto Matematico, Pisa, Italy*

Let n and k be integers, $n \ge 2$, and $1 \le k \le n-1$. Hoggatt has recently noted that

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

Gould [1] conjectures that

$$\mathrm{GCD}\left(\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right) = \mathrm{GCD}\left(\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right).$$

In this note, I shall prove this conjecture and obtain the corollary that these GCD's are equal to

$$\mathrm{GCD}\left(\binom{n-1}{k-2}, \binom{n-1}{k-1}, \binom{n-1}{k}, \binom{n-1}{k+1}\right).$$

Before proceeding, let us note that the six binomial coefficients involved form a hexagon about $\binom{n}{k}$ in the Pascal triangle. The two groups of three involved are the two equilateral triangles of this hexagon.

<u>Theorem</u>. For $n \ge 2$, and $1 \le k \le n-1$, we have that

$$\mathrm{GCD}\left(\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right) = \mathrm{GCD}\left(\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right).$$

<u>Proof</u>. Let the two GCD's be G_1 and G_2, respectively. We write out the involved section of the Pascal triangle as:

$$\begin{matrix} a & & b & & c & & d \\ & a+b & & b+c & & c+d & \\ & & a+2b+c & & b+2c+d & & \end{matrix},$$

where $b + c = \binom{n}{k}$, etc. Then $G_1 = \mathrm{GCD}(b, c+d, a+2b+c)$ and $G_2 = \mathrm{GCD}(c, a+b, b+2c+d)$. (If $k = 1$ (or $k = n-1$) then $a = 0$ (or $d = 0$). The following argument still holds in these cases, but one can see that $G_1 = G_2 = 1 = \mathrm{GCD}(a, b, c, d)$ directly.)

We shall show that $p^e \mid G_1$ if and only if $p^e \mid G_2$, for any prime power p^e.

*This work was supported by a research fellowship of the Italian National Research Council.

Case 1. If $p^e \mid b$ and $p^e \mid c$, then $p^e \mid G_1$ iff $p^e \mid (a,d)$ iff $p^e \mid G_2$.

Case 2. If $p^e \nmid b$ and $p^e \nmid c$, then $p^e \nmid G_1$ and $p^e \nmid G_2$.

Case 3. If $p^e \mid b$ and $p^e \nmid c$, then $p^e \nmid G_2$. Suppose that $p^e \mid G_1$. Then we have $p^e \mid c + d$ and $p^e \mid a + c$, whence $p^e \nmid a$ and $p^e \nmid d$. We claim that the four conditions $p^e \nmid a$, $p^e \mid b$, $p^e \nmid c$ and $p^e \mid c + d$ are inconsistent. For this we require a lemma.

Lemma. For $0 \le k < n$,

$$p^e \Big| \binom{n}{k} \quad \text{and} \quad p^e \nmid \binom{n}{k+1}$$

implies $p \mid k + 1$.

Proof. Let $n = \Sigma a_i p^i$ and $b = \Sigma b_i p^i$ be the p-ary expansions of n and k. A result of Glaisher [2, Corollary 6.1] asserts that

$$p^\alpha \Big\| \binom{n}{k}$$

if and only if α is the number of borrows in the p-ary subtraction $n - k$. Consider now b_0 and a_0. If $0 \le b_0 < a_0$ or $a_0 < b_0 < p - 1$, then $n - k$ and $n - (k + 1)$ have the same number of borrows. If $b_0 = a_0 < p - 1$, then $n - (k + 1)$ has more borrows than $n - k$. Hence $b_0 = p - 1$ is the only case consistent with

$$p^e \Big| \binom{n}{k} \quad \text{and} \quad p^e \nmid \binom{n}{k+1}.$$

Corollary. For $0 \le k < n$,

$$p^e \nmid \binom{n}{k} \quad \text{and} \quad p^e \Big| \binom{n}{k+1}$$

implies $p \mid n - k$.

Proof. Use $\binom{n}{k} = \binom{n}{n-k}$ and the Lemma.

Returning to the Theorem, we have

$$p^e \nmid \binom{n-1}{k-2} = a \quad \text{and} \quad p^e \Big| \binom{n-1}{k-1} = b\,,$$

hence $p \mid n - k + 1$, and we have

$$p^e \Big| \binom{n-1}{k-1} = b \quad \text{and} \quad p^e \nmid \binom{n-1}{k} = c\,,$$

hence $p \mid k$. Thus $p \mid n + 1$. Now $c + d = \binom{n}{k+1}$. Let $n = \Sigma a_i p^i$ and $k + 1 = \Sigma b_i p^i$ be the p-ary expansions. From $p \mid n + 1$, we have $a_0 = p - 1$ and from $p \mid k$, we have $b_0 = 1$. Hence $n - (k + 1)$ has the same number of borrows as $(n - 1) - k$. From Glaisher's result and

$$p^e \nmid \binom{n-1}{k} = c\,,$$

we deduce that $p^e \nmid \binom{n}{k+1} = c + d$, which demonstrates the claimed inconsistency. Thus, in Case 3, $p^e \nmid G_1$ and $p^e \nmid G_2$.

Case 4. If $p^e \nmid b$ and $p^e \mid c$, then the symmetry of the binomial coefficients converts this to Case 3 and this completes the theorem.

Corollary. $G_1 = G_2 = \text{GCD}(a,b,c,d)$.

Proof. We have $G_1 = G_2$ from the Theorem and so we have $G_1 \mid b$, $G_1 \mid c$, $G_1 \mid c + d$, $G_1 \mid d$, $G_1 \mid a$ and $G_1 \mid GCD(a,b,c,d)$. Conversely, $GCD(a,b,c,d)$ clearly divides G_1.

REFERENCES

1. H. W. Gould, "A New Greatest Common Divisor Property of the Binomial Coefficients," Notices Amer. Math. Soc., 19 (1972) A-685, Abstract 72T-A248.
2. D. Singmaster, Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers, to appear.

LETTERS TO THE EDITORS

Dear Editors:

On page 165 of Professor Coxeter's Introduction to Geometry (New York, 1961), we read: "In 1202, Leonardo of Pisa, nicknamed Fibonacci ("son of good nature"), came across his celebrated sequence ···."

This translation of Leonardo's nickname differs, of course, from the one I've seen in the Quarterly.

Who can solve the historic mystery for us?

Les Lange
Dean, School of Science
San Jose State University
San Jose, California

Dear Editors:

Thank you for the reprints I have just received. Sorry to bother you again, but somehow the main sentence from "An Old Fibonacci Formula and Stopping Rules," (Vol. 10, No. 6) was omitted. The formula is

$$\sum_{0}^{\infty} \frac{F(n)}{2^{n+1}} = 1$$

and it is based on Wald's proof that the defined stopping rule is a real stopping rule (the process terminates after a final number of steps with probability 1).

R. Peleg
Jerusalem, Israel