A DISCUSSION OF SUBSCRIPT SETS WITH SOME FIBONACCI COUNTING HELP

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1. INTRODUCTION

If the elements of continued fraction-oriented physical and mathematical systems are systematically arranged with respect to subscripts attached to the elements, the choice of order and parity for the subscripts often leads to easily implemented algorithms for the combinatorial determination of the subscripts. All the essential information of the problem can be carried by the subscripts since integer manipulation of the subscripts can substitute for algebraic manipulation of the elements of the system. Specific and general sets of subscripts are discussed, together with the application of Fibonacci methods for the counting of members of subscript sets.

2. "BASIC" SUBSCRIPT SETS AND THEIR GENERATION

The Euler-Minding formulas are introduced early in Perron's classic "Die Lehre von den Kettenbrüchen" [1] and figure prominently in much of the subsequent continued fraction discussions. If Perron's notation is altered slightly to eliminate (for convenience) the zero subscript, the Euler-Minding formulas appear as

$$\text{(1)}\quad \begin{aligned} S_n = a_1a_2\cdots a_n\Bigg(1 &+ \sum_{j}^{1,n-1}\frac{c_j}{a_ja_{j+1}} + \sum_{j<k}^{1,n-2}\frac{c_j}{a_ja_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}} \\ &+ \sum_{j<k<\ell}^{1,n-3}\frac{c_j}{a_ja_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}}\,\frac{c_\ell}{a_{\ell+2}a_{\ell+3}} + \cdots\Bigg) \end{aligned}$$

$$\text{(2)}\quad \begin{aligned} T_{n-1} = a_2a_3\cdots a_n\Bigg(1 &+ \sum_{j}^{k,n-2}\frac{c_j}{a_ja_{j+1}} + \sum_{j<k}^{1,n-3}\frac{c_j}{a_ja_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}} \\ &+ \sum_{j<k<\ell}^{1,n-4}\frac{c_j}{a_ja_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}}\,\frac{c_\ell}{a_{\ell+2}a_{\ell+3}} + \cdots\Bigg) . \end{aligned}$$

There are $\left[\frac{n}{2}\right]$ summations plus the one in the parentheses of (1) and $\left[\frac{n-1}{2}\right]$ summations plus the one in the parentheses of (2).*

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*The brackets specify the largest integer less than or equal to the number bracketed.

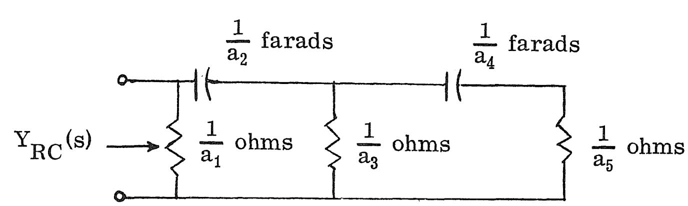
By letting the c's of (1) and (2) assume particular values, the ratio S_n/T_{n-1} can be used to describe various rational fraction forms of continued fractions some of which are directly related to physical structures. For example, if the c's are all equal to one, the ratio S_n/T_{n-1} is the rational fraction equivalent of the continued fraction [1]

(3) $$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +} \cdots + \frac{1}{a_n} = \frac{S_n}{T_{n-1}}\,.$$

More concretely, for n equal five,

(4) $$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\,\frac{1}{a_5} = \frac{a_5a_4a_3a_2a_1 + (a_5a_4a_3 + a_5a_4a_1 + a_5a_2a_1 + a_3a_2a_1) + (a_5 + a_3 + a_1)}{a_5a_4a_3a_2 + (a_5a_4 + a_5a_2 + a_3a_2) + 1}\,.$$

Salzer [2] in an interpolation problem sets all c's equal to $(x - x_i)$ in the ratio S_n/T_{n-1} and uses the continued fraction process to retrieve $a_1, a_2, \cdots$. As a further example, by letting the c's equal the complex frequency variable $s = \sigma + j\omega$, the impedance or admittance of two-element kind electrical ladder networks can be described by S_n/T_{n-1}. For instance, the resistance-capacitance network



has the S_n/T_{n-1} ratio [3]

(5) $$Y_{RC}(s) = \frac{(a_5 + a_3 + a_1)s^4 + (a_5a_4a_3 + a_5a_4a_1 + a_5a_2a_1 + a_3a_2a_1)s^2 + a_5a_4a_3a_2a_1}{s^4 + (a_5a_4 + a_5a_2 + a_3a_2)s^2 + a_5a_4a_3a_2}\,.$$

It is seen that the ascending subscript arrangement in the continued fraction of (4) and in the physical network above both lead to rational fractions having numerators and denominators with sums of products of n or less coefficients with the sums of products of <u>no</u> coefficients being interpreted as the numeric one. Features immediately apparent with each sum of h coefficients are the lexicographical order of subscripts, the absence of repeats, and the presence of a leading $a_na_{n-1} \cdots a_{n+1-h}$ and a final $a_ha_{h-1} \cdots a_1$ or $a_{h+1}a_h \cdots a_2$.

It is seen that equationwise all the information needed for the construction of the rational fraction is contained in the subscripts alone.

The subscripts of the coefficients of a sum of products of h coefficients thus constitute a subscript set. The numerator and denominator of the rational fraction can thereby be represented as a collection of subscript sets. Because of the basic nature of (3) and because of the basic role played by the subscripts exemplified by (4) in specifying properties of more general subscript sets, the subscripts of a sum of products of h coefficients determined from a continued fraction as in (3) are called basic subscript sets and are given the symbol $\{N_n^h\}_0$ where n is the largest subscript of the set, h is the number coefficients in each product, and the 0 subscript on the braces identifies the set as "basic." A typical basic subscript set from (4) is $\{5, 4, 3;\ 5, 4, 1;\ 5, 2, 1;\ 3, 2, 1\}$.

What are the precise properties of basic subscript sets? How can they be generated easily, and what is the power of a basic subscript set? A discussion follows.

Consider a sequence of h non-zero, non-repeating integers, called subscripts. The subscripts in the sequence are arranged in alternating parity and descending size with the largest subscript (on the left) assigned a specific parity. A basic subscript set has as members all possible such sequences with the largest subscript in any sequence not exceeding n. The subscript sets are represented as

$$\{N_n^h\}_0 = \boxed{;}_{f=0}^{n/2} (n - 2f), \quad \{N_{h-2f-1}^{h-1}\}, \qquad n \text{ even}, \tag{6}$$

$$\{N_n^h\}_0 = \boxed{;}_{f=0}^{(n-1)/2} (n - 2f), \quad \{N_{h-2f-1}^{h-1}\}, \qquad n \text{ odd}. \tag{7}$$

$\{N_n^0\}_0$ stands for no subscripts and is associated with the numeric one or a single term with no coefficients. (See, for example, the denominators of (4) and (5).) $\{N_n^k\}_0$ for $k > n$ is the null set with no value. The boxed semicolon $\boxed{;}$ is a symbol for collecting the sequences of a subscript set.

If n is odd (even), the largest subscript of any sequence has odd (even) parity. The smallest subscript of any sequence has odd (even) parity if $n - h + 1$ is odd (even).

From (6) and (7), it can be determined that the starting* sequence-last sequence pair of $\{N_n^h\}_0$ assume either (8) and (9) or (10) and (11).

$$\left.\begin{array}{ll} (8) & n, n-1, n-2, \cdots, n-h+1 \\ (9) & h, h-1, h-2, \cdots, 1 \end{array}\right\} \quad n - h + 1 \text{ odd}$$

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*No other sequence with the prescribed properties can be found which has a larger subscript in a given position than the subscript in that position in the starting sequence. If "less than" is substituted for "larger than," the last sequence is described.

$$\left.\begin{array}{ll} (10) \quad & n,\ n-1,\ n-2,\ \cdots,\ n-h+1 \\ (11) \quad & h+1,\ h,\ h-1,\ \cdots,\ 2 \end{array}\right\},\quad n-h+1 \text{ even}.$$

Note that the difference between given position subscripts in the starting and last sequences is a constant q, where $q = (n - h)$ for $(n - h) + 1$ odd and $q = (n - h - 1)$ for $(n - h) + 1$ even. In either case, q is even. This is a property which is valid for the more general subscript sets discussed later.

An algorithm to generate basic subscript sets can be deduced from an inspection of (1) and (2) once the starting and last sequence have been established. Assume that the f^{th} member of a subscript set is known. To find the $(f + 1)^{st}$ member, start at the right side of the f^{th} member and scan the subscripts toward the left until the first subscript is found which has a value of at least two greater than the corresponding position subscript of the last sequence. Subtract two from this subscript to obtain the subscript for the $(f + 1)^{st}$ member and complete the $(f + 1)^{st}$ member by filling all positions to the right with the largest possible subscripts consistent with size-order and position parity. Note that subtraction of two's is necessary to retain position parity.

The implementation of the algorithm is even simpler than the description as is illustrated in the "by hand" generation of $\{N_8^4\}_0$ in (12).

$$(12)\qquad \begin{array}{ccccc}
8,\ 7,\ 6,\ 5 & 8,\ 7,\ 4,\ 3 & 8,\ 5,\ 4,\ 3 & 8,\ 3,\ 2,\ 1 & 6,\ 5,\ 2,\ 1 \\
\underline{\qquad\quad -2} & \underline{\qquad\quad -2} & \underline{\qquad\quad -2} & \underline{-2\qquad\quad} & \underline{\quad -2\qquad} \\
8,\ 7,\ 6,\ 3 & 8,\ 7,\ 4,\ 1 & 8,\ 5,\ 4,\ 1 & 6,\ 5,\ 4,\ 3 & 6,\ 3,\ 2,\ 1 \\
\underline{\qquad -2\quad} & \underline{\qquad -2\quad} & \underline{\qquad -2\quad} & \underline{\qquad\quad -2} & \underline{-2\qquad\quad} \\
8,\ 7,\ 6,\ 1 & 8,\ 7,\ 2,\ 1 & 8,\ 5,\ 2,\ 1 & 6,\ 5,\ 4,\ 1 & 4,\ 3,\ 2,\ 1 \\
\underline{\quad -2\qquad} & \underline{\quad -2\qquad} & \underline{\quad -2\qquad} & \underline{\quad -2\qquad} & \\
8,\ 7,\ 4,\ 3 & 8,\ 5,\ 4,\ 3 & 8,\ 3,\ 2,\ 1 & 6,\ 5,\ 2,\ 1 &
\end{array}$$

What is the power of a basic subscript set? It can be shown by comparison with a physical model that the power of the collection of either numerator subscript sets or denominator subscript sets is Fibonaccian and this, in turn, provides a clue to the answer.

It is well established [4]-[6] that the resistance or conductance of electrical ladder networks has as the ratio of numerator terms to denominator terms a ratio of Fibonacci numbers. For example, if a ladder network is composed of $\underline{n}$ unit conductances with a shunt conductance at the input end and either a shunt conductance ($\underline{n}$ odd) or a short circuit ($\underline{n}$ even) at the output end, the conductance measured at the input terminals is given by*

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*Several other forms in terms of resistance or conductance are, of course, possible. For example, Basin [6] states the input resistance of the dual of the above network with $\underline{n}$ even as F_{2n+1}/F_{2n}. However, Basin's $\underline{n}$ is half the n of this paper because of a choice in size of his unit network.

(13) $$G_n = \frac{F_{n+1}}{F_n} \text{ mhos },$$

where $F_1, F_2, F_3, F_4, \cdots = 1, 1, 2, 3, \cdots$ are the well-known Fibonacci numbers. Moreover, if the shunt arms of the ladder network are replaced and described by odd subscripted admittances (y's) and the series arms are replaced and described by even subscripted impedances (z's) with the numbering increasing away from the input terminals, (4) exemplifies the continued fraction and rational fraction form of the input admittance. To complete the identification, odd subscripted a's of (4) are interpreted as y's, and even subscripted a's are interpreted as z's. It can be seen that the power of a collection of basic subscript sets is given by

(14)
$$\begin{aligned}
\left|\{N_n^{n-1}\}_0 + \{N_n^{n-3}\}_0 + \cdots + \{N_n^0\}_0\right| &= \left|\{N_n^{n-1}\}_0\right| + \left|\{N_n^{n-3}\}_0\right| + \cdots + \left|\{N_n^0\}_0\right| \\
&= \left|\{N_{n-1}^{n-1}\}_0 + \{N_{n-1}^{n-3}\}_0 + \cdots + \{N_{n-1}^0\}_0\right| \\
&= \left|\{N_{n-1}^{n-1}\}_0\right| + \left|\{N_{n-1}^{n-1}\}_0\right| + \cdots + \left|\{N_{n-1}^0\}_0\right| = F_n, \quad n \text{ odd},
\end{aligned}$$

(15)
$$\begin{aligned}
\left|\{N_n^{n-1}\}_0 + \{N_n^{n-3}\}_0 + \cdots + \{N_n^1\}_0\right| &= \left|\{N_n^{n-1}\}_0\right| + \left|\{N_n^{n-3}\}_0\right| + \cdots + \left|\{N_n^1\}_0\right| \\
&= \left|\{N_{n-1}^{n-1}\}_0 + \{N_{n-1}^{n-3}\}_0 + \cdots + \{N_{n-1}^1\}_0\right| \\
&= \left|\{N_{n-1}^{n-1}\}_0\right| + \left|\{N_{n-1}^{n-3}\}_0\right| + \cdots + \left|\{N_{n-1}^1\}_0\right| = F_n, \quad n \text{ even}.
\end{aligned}$$

That $\{N_n^h\}_0$ might be equal to a Fibonacci-related binomial coefficient is suggested in a paper by Raab [9] in this Journal. Raab shows that by selecting the entries of a certain diagonal of the Pascal triangle array, the Fibonacci numbers are given by

(16) $$F_n = \sum_{\delta=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-\delta}{\delta}.$$

However, Perron [1] lists term-by-term the identical binomial coefficients obtained in the expansions of (1) and (2). This verifies, as was suspected, that

(17) $$\left|\{N_n^h\}_0\right| = \binom{\left[\frac{n+h}{2}\right]}{\left[\frac{n+h}{2}\right] - h}.$$

3. GENERAL SUBSCRIPT SETS

It is apparent that the basic subscript sets belong to a more general class of subscript sets. Consider a set of all possible sequences of h, non-zero, non-repeating, positive integers called subscripts, having the properties that no subscript exceeds M or is less than m and that each sequence within a set has the same parity order. Let it be further specified that each sequence be arranged in descending size order from left to right. Thus, there is a unique starting sequence and a unique last sequence. The leftmost position of the starting sequence is occupied by a subscript $\leq M$ (depending on mutual parities), and the remaining $(h - 1)$ positions are filled with the largest subscripts possible consistent with size-order and parity. Similarly, the rightmost position of the last sequence is occupied by a subscript $\geq m$ (depending on mutual parities), and the remaining $(h - 1)$ positions are occupied by the smallest consistent subscripts. For example, if $h = 6$, $M = 20$, $m = 3$ and position parity is even, odd, even, even, even, even, the starting sequence must be 20, 19, 18, 16, 14, 12, and the last sequence must be 12, 11, 10, 8, 6, 4. Because the position parity must be the same for the starting and last sequence and because of the compacting of subscripts to the left in the starting sequence and to the right in the last sequence, the difference between the same position subscripts within the starting and last sequences is the same. From this fact, it can be seen that there is a constant difference q between corresponding position subscripts in the starting and last sequences, and moreover, q must be even as the result of position parity. Once a starting and last sequence are determined, the generation of subscript sets in general follows the algorithm given for basic subscript sets. Of course, parity must be strictly observed.

While (17) applies in particular to basic subscript sets and is useful for counting them without first determining the starting and last sequences, it is possible to use (17) to obtain a new form suitable for counting all subscript sets.

Consider $\left|\{N_n^h\}_0\right|$. If n and h are both odd or both even (i.e., $n + h$ is even),

$$\left[\frac{n+h}{2}\right] = \frac{n+h}{2} \quad . \tag{18}$$

Since the last member of the starting sequence is $(n - h + 1)$, it must be odd. This makes q

$$q = (n - h + 1) - 1 = (n - h) \, . \tag{19}$$

If n is odd and h even or vice versa (i.e., $n + h$ is odd),

$$\left[\frac{n+h}{2}\right] = \frac{n+h-1}{2} \quad . \tag{20}$$

In this case, the value of q is

$$q = (n - h + 1) - 2 = (n - h - 1) \, . \tag{21}$$

Elimination of n between either (18) and (19) or between (19) and (20) results in the single equation

$$\{N_n^h\}_0 = \binom{h + q/2}{q/2} \tag{22}$$

which is independent of n and the parity of $(n + h)$.

Next, consider the sequences of differences between any sequence and the last sequence of $\{N_n^h\}_0$. This set of differences starts with a sequence of $\underline{h}$ $\underline{q}$'s, $(q, q, q, \cdots, a)$ and ends with the sequence of $\underline{h}$ zeros $(0, 0, 0, \cdots, 0)$. The same algorithm applied to the sequence of differences produces members of the difference set in one-to-one correspondence with the members of the basic subscript set, and thereby (22) is applicable for counting them. However, a little reflection reveals that the same $(q, q, q, \cdots, q)$ to $(0, 0, 0, \cdots, 0)$ sequences apply to any subscript set having the given q and h. Thus, (22) can be recast more generally as

$$R_{h,q} = \binom{h + q/2}{q/2}. \tag{23}$$

4. SOME USEFUL NON-BASIC SUBSCRIPT SETS

It was noted earlier that $\{N_n^h\}_0$ provided subscripts for a sum of products of coefficients such as $a_x a_y a_z \cdots$ (see (4)). If the even subscripted a's represent one kind of item (as in (5)) and the odd subscripted a's represent another, the sequences of the basic subscript set represent sums of products of kinds of things in a fixed alternation pattern. For example, in another of the physical systems described earlier, the odd subscripted a's were shunt arm admittances (y's) and the even subscripted a's were series arm impedances (z's). In the case of a lumped element ladder network, a product has a specific $\cdots zyz \cdots$ order. In the study of certain cascaded distributed element transmission systems, a mathematical interaction takes place which, in effect, keeps the $\cdots zyz \cdots$ order the same but introduces additional sums of products in which even subscript positions replace some or all of the formerly odd subscript positions of the basic subscript set [10], [11].

Let $\{N_n^h\}_\ell$ be a subscript set whose subscripts describe the same element product order as is described by the basic subscript set but whose sequences each have ℓ of the odd subscript positions of $\{N_n^h\}_0$ replaced by ℓ even subscript positions. If g is the number of odd parity positions in a sequence of $\{N_n^h\}_0$, there are $\binom{g}{\ell}$ distinct types of parity arrangement for the sequences of $\{N_n^h\}_\ell$. To obtain $\{N_n^h\}$, it is feasible to form $\binom{g}{\ell}$ subsets each having its own starting sequence and last sequence. The subsets are designated $\{N_n^h\}_{\ell_1}$, $\{N_n^h\}_{\ell_2}$, etc., and are generated and or counted just like any subscript set. Let the position of the rightmost odd subscript of $\{N_n^h\}_0$ be designated odd position 1, next on the left odd position 2, etc., up to and including g. Determine the names of the $\binom{g}{\ell}$ combinations of the odd position numbers $1, 2, \cdots, g$ taken ℓ at a time. For each combination of odd position numbers, the sequences of the subsets have the parity arrangement of $\{N_n^h\}_0$ except for ℓ former odd subscript positions replaced by ℓ even subscript positions. The subscripts of the starting sequence should be as large as consistently possible and those of the last sequence as small as consistently possible. While the power of the individual subsets can be found from (23), the power of $\{N_n^h\}_\ell$ is given by

$$\left|\{N_n^h\}_\ell\right| = \sum_{i=1}^{\binom{g}{\ell}} \binom{h + \frac{q_i}{2}}{\frac{q_i}{2}} \qquad (24)$$

5. DERIVATION OF ℓ_{MAX}

For the physical systems which utilize $\{N_n^h\}_\ell$, the value of ℓ_{max} for each h is of great use in determining the number of coefficients, and hence size, of governing equations. Certainly ℓ_{max} cannot exceed g and there are many possible situations in which ℓ_{max} cannot even equal g. It is shown below, in fact, that ℓ_{max} is equal to the lesser of q/2 or g of $\{N_n^h\}_0$.

The starting and last sequences, respectively, of $\{N_n^h\}_0$ take on either of the two forms given by (8), (9) or (10), (11). Since corresponding position subscripts are of the same parity, n and h in (8) and (9) can be either both even or both odd. In (10) and (11), if n is even, h is odd, and if n is odd, h is even.

(a) n, h both even (Eqs. (8) and (9)). There are h/2 even and h/2 = g odd subscripts in any sequence. If $n \geq 2h$, there are exactly (equals sign) or more than h even subscripts available between n and 1 (including n). Thus, if n - h = q is divided by two, and thereby $q/2 \geq h/2$, a sequence with all even subscripts can be found. Thus ℓ_{max} is not limited by q/2 since h/2 odd positions have been filled with even subscripts. If $n < 2h$, there are less than h even subscripts available between n and 1 (including n). This is reflected by $q/2 < h/2$. The value for ℓ_{max} must be q/2.

(b) n, h both odd (Eqs. (8) and (9)). There are (h - 1)/2 even and (h + 1)/2 = g odd subscripts in any sequence. If $n \geq 2h + 1$, there are exactly (equals sign) or more than h odd subscripts between n and 1 (exclusive of 1). Thus, if $q/2 \geq (h + 1)/2$, there are at least h odd subscripts between n and 1 (exclusive of 1) which can be reduced by one to give at least h even subscripts. Such a sequence would have (h + 1)/2 former odd positions filled by even subscripts. Therefore, ℓ_{max} is not limited by q/2 since (h + 1)/2 odd positions have been filled by even subscripts. If $q/2 < (h + 1)/2$, the value for ℓ_{max} must be q/2.

(c) n even, h odd (Eqs. (10) and (11)). There are (h + 1)/2 even and (h - 1)/2 = g odd subscripts in any sequence. If $n \geq 2h$, there are h distinct even subscripts between n and 2 (including n and 2). The condition can be arranged as $n - 1 \geq 2h - 1$ or $n - 1 - h \geq h - 1$ or $(n - 1 - h)/2 \geq (h - 1)/2$, where $(n - 1 - h) = q$. Since fulfillment of this condition fills (h - 1)/2 odd positions with even subscripts, ℓ_{max} is not limited by q/2. If $q/2 < (h - 1)/2$, the value for ℓ_{max} must be q/2.

(d) n odd, h even (Eqs. (10) and (11)). There are h/2 even and h/2 = g odd subscripts in any sequence of the basic set. If $n \geq 2h + 1$ there are exactly (equals sign) or more than h odd subscripts between n and 2 (including n) which can be reduced by one to give at least h even subscripts. Therefore, $n - h - 1 \geq h$, $(n - h - 1)/2 \geq h/2$, and ℓ_{max} is not limited by $q/2$. If $q/2 < h/2$, the value for ℓ_{max} must be $q/2$.

From (a), (b), (c), and (d), it is seen that in all cases $q/2$ is the value for ℓ_{max} if $q/2$ is less than or equal to g, the number of odd positions in a sequence, and g is the value for ℓ_{max} if $q/2$ is greater than or equal to g. A sufficient condition for $q/2$ to be the greatest ℓ_{max} for a given n and <u>any</u> h occurs when $q/2 = g$.

6. EXAMPLE OF $\{N_n^h\}_\ell$

$\{N_8^4\}_0$			$\{N_8^4\}_{1,1}$	$\{N_8^4\}_{1,2}$	$\{N_8^4\}_2$
8765	8721	6543	8764	8643	8642
8763	8543	6521	8762	8641	
8761	8541	6521	8742	8621	
8743	8521	6321	8542	8421	
8741	8321	4321	6542	6421	

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