# ANALYTICAL VERIFICATION OF AN "AT SIGHT" TRANSFORMATION

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There has been interest recently [1]-[3] in finding easy, at sight ways of transforming

(1) $$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

to

(2) $$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 = 0$$

subject to

(3) $$z = (s+1)/(s-1)\,.$$

Power [1], [2] discovered an $(n+1) \times (n+1)$ X matrix applicable for transforming the a's of (1) to the b's of (2), and Fielder [3] developed a somewhat different Q matrix for the same purpose. In each case, the most rewarding feature of the <u>application</u> matrix (X or Q) was the apparent presence of an extremely simple combinatorial scheme for constructing the application matrix from the barest minimum of information. While many trial values of n assured that either application matrix was safe to use for practical purposes, the analytic validity of the combinatorial schemes defied verification, leaving an understandably disturbing theoretical situation.

In this note, a very simple proof of the validity of the combinatorial method for constructing the Q matrix is presented. The proof is effected through application of generating functions.*

It has been shown [3] that the coefficients of (1) and (2) are related through

(4) $$b = \frac{1}{A_0} Q a\,,$$

where a is the $(n+1) \times 1$ column matrix $(1, a_{n-1}, \cdots, a_0)$, b is the $(n+1) \times 1$ column matrix $(1, b_{n-1}, \cdots, b_0)$, Q is an $(n+1) \times (n+1)$ application matrix of integers, and $A_0$ is the sum of the elements of a. It has also been shown [3] that Q is the product of two $(n+1) \times (n+1)$ matrices of integers PN. Briefly, the elements of P and N are given, respectively, by

(5) $$p_{i,j} = (-1)^{n-i}(-2)^{j-1}\binom{n-j+1}{n-i+1}, \qquad (i,j = 1, 2, \cdots, n+1)\,,$$

*In private correspondence with Fielder, Power outlines an analytic verification of the combinatorial process for constructing his X matrix. The proof presented herein for the Q matrix is an independent development and is, of course, different from Power's.

(6) $$\nu_{i,j} = \binom{n - j + 1}{i - 1}, \qquad (i,j = 1, 2, \cdots, n + 1).$$

As a brief digression, $Q_5$ is exemplified to illustrate the conjectured combinatorial construction method.

(7) $$Q_5 = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{array}\right]$$

The conjectured combinatorial pattern is

(8) $$q_{i+1,j+1} + q_{i,j+1} + q_{i,j} = q_{i+1,j}, \qquad (i,j = 1, 2, \cdots, n)$$

and is illustrated for $10 - 5 - 3 = 2$. Once (8) is established, all that is needed to assure a valid $Q$ matrix in general is a first row of $(n + 1)$ ones and a last column of alternating sign binomial coefficients for the appropriate index $n$.

From a consideration of (5) and (6), it is seen that the general element of $Q$ is given by

(9) $$q_{i,j} = (-1)^{i-1} \sum_{k=1}^{\rho_{i,j}} (-2)^{k-1} \binom{n - k + 1}{n - i + 1} \binom{n - j + 1}{k - 1}, \qquad (i,j = 1, 2, \cdots, n + 1).$$

Since $P$ has all zero elements above the main diagonal with non-zero elements elsewhere, and $N$ has all-zero elements below the secondary diagonal with non-zero elements [3], it can readily be seen that the upper summation index $\rho_{i,j}$ of (9) is the lesser of the number of non-zero elements of row $i$ of $P$ or column $j$ of $N$. This value can be established as

(10) $$\rho_{i,j} = \tfrac{1}{2}(n + 1 + i - j - |n + 2 - i - j|).$$

In any event, if $\rho_{i,j}$ is replaced by $(n + 1)$ the laws of matrix multiplication assure that $q_{i,j}$ so determined is the correct value for the element of $Q$. Subsequent work, however, is based on $q_{i,j}$ having the correct value as the upper summation index approaches infinity. This extension is covered by Lemma 1.

<u>Lemma 1.</u>

(11) $$q_{i,j} = (-1)^{i-1} \sum_{k=1}^{\infty} (-2)^{k-1} \binom{n - k + 1}{n - i + 1} \binom{n - j + 1}{k - 1}, \qquad (i,j = 1, 2, \cdots, n + 1).$$

<u>Proof.</u> For an upper index of $(n + 1)$, $q_{i,j}$ is valid. For $k > (n + 1)$, the upper term of the first binomial coefficient of (11) is negative. The value of this binomial coefficient is thereby zero. Thus, for any upper index greater than $(n + 1)$ no contribution to $q_{i,j}$ is

made.* Q. E. D.

Lemma 2. Given the generating functions

$$G_j = \sum_{i=1}^{\infty} g_{i,j} x^{i-1}, \qquad G_{j+1} = \sum_{i=1}^{\infty} g_{i,j+1} x^{i-1} . \tag{12}$$

Then,

$$g_{i+1,j+1} + g_{i,j+1} + g_{i,j} = g_{i+1,j} \tag{13}$$

iff

$$G_{j+1} = \frac{1 - x}{1 + x} G_j .$$

Proof. For necessity, assume (13) is true. A relation for the generating functions is

$$G_{j+1} + xG_{j+1} + xG_j = G_j , \tag{15}$$

from which it is seen that (14) follows. For sufficiency, assume (14) true. Form (15) and equate coefficients according to (12) to establish (13) in general.

The following theorem verifies (8).

Theorem. With matrices P and N as defined above, Q = PN has elements $q_{i,j}$ which satisfy

$$q_{i+1,j+1} + q_{i,j+1} + q_{i,j} = q_{i+1,j}, \qquad (i,j = 1, 2, \cdots, n + 1) . \tag{16}$$

Proof. Let W be a $1x(n + 1)$ row matrix

$$W = (1, x, x^2, \cdots, x^n) . \tag{17}$$

Then, WP is a row matrix whose $j^{th}$ element is the generating function for the $j^{th}$ column of P, i.e., the $(n + 1) \times 1$ column vectors of P, the powers of x increasing down the column.

$$WP = \left\{ (-2)^{j-1} \sum_{k=1}^{n+1} (-1)^{k-2} \binom{n - j + 1}{n - k + 1} x^{k-1} \right\} = \left\{ (2x)^{j-1}(1 - x)^{n-j+1} \right\} . \tag{18}$$

Because of the associative property of matrix multiplication, (WP)N = W(PN) = WQ. Hence, with use of Lemma 1

$$\begin{aligned}(WP)N &= \left\{ \sum_{k=1}^{\infty} \binom{n - j + 1}{k - 1} (2x)^{k-1}(1 - x)^{n-k+1} \right\} \\ &= \left\{ (1 - x)^{j-1} \sum_{k=1}^{\infty} \binom{n - j + 1}{k - 1} (2x)^{k-1}(1 - x)^{n-k-j+2} \right\} \\ &= \left\{ (1 - x)^{j-1}([1 - x] + 2x)^{n-j+1} \right\} = \left\{ (1 - x)^{j-1}(1 + x)^{n-j+1} \right\} \\ &= WQ = \left\{ Q_j \right\} , \qquad (j = 1, 2, \cdots, n + 1).\end{aligned} \tag{19}$$

*Note that for either i or j (or both) greater than n + 1, the value $q_{i,j}$ is zero regardless of the upper index.

But WQ is a row matrix whose elements are the generating functions of the column elements of Q. From (19), it is seen that

$$Q_{j+1} = (1 - x)^j (1 + x)^{n-j} = \frac{1 - x}{1 + x} Q_j . \tag{20}$$

Thus, by Lemma 2,

$$q_{i+1,j+1} + q_{j,j+1} + q_{i,j} = q_{j+1,j}, \quad (i,j = 1, 2, \cdots, n + 1) . \tag{21}$$

Q.E.D.

Corollary to the Theorem. The matrix $Q = \{q_{i,j}\}$, $(i, j = 1, 2, \cdots, n+1)$ is such that

$$q_{1,j} = 1 \qquad (j = 1, 2, \cdots, n + 1) , \tag{22}$$

$$q_{i,n+1} = (-1)^{i-1}\binom{n}{n - i + 1} \qquad (i = 1, 2, \cdots, n + 1) . \tag{23}$$

Proof. Since $Q_j = (1 - x)^{j-1}(1 + x)^{n-j+1}$, it follows that the constant term of the generating function is always unity regardless of j. This proves (22). It can be seen from (20) that

$$Q_{n+1} = (1 - x)^n . \tag{24}$$

The coefficients of the generating functions are thereby identically those values given by (23).

The establishment of specific forms for (22) and (23) means that the first row and last column can be immediately written down, and the remainder of the elements of Q follow from application of the combinatorial rule.

As a check on the operational calculations, the bottom row of Q starting with the left element should consist of $1, -1, 1, -1, \cdots$ and the leftmost column starting with the upper element should be the coefficients of increasing powers of x in $(1 + x)^{n+1}$.

## REFERENCES

1. H. M. Power, "The Mechanics of the Bilinear Transformation," IEEE Trans—Education (Short Papers), Vol. E-10, pp. 114-116, June, 1967.
2. H. M. Power, "Comments on 'The Mechanics of the Bilinear Transformation'," IEEE Trans. Education (Correspondence), Vol. E-11, p. 159, June 1968.
3. D. C. Fielder, "Some Classroom Comments on Bilinear Transformations," submitted to IEEE Trans. on Education.