COMPLETE SEQUENCES OF FIBONACCI POWERS

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1. INTRODUCTION

We define the Fibonacci sequence in the usual manner by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for each k in the set N of positive integers. An integer is said to have a representation with respect to a sequence if it is the sum of some finite subsequence. A sequence is complete if every positive integer has a representation with respect to the sequence.

In [2], Hoggatt and King showed that the sequence $\{F_k\}$ of Fibonacci numbers is complete, and O'Connell showed in [3] that the sequence 1, 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ of two of each of the Fibonacci squares F_k^2 is complete. In Theorem 1 of this paper we show that the sequence of 2^{n-1} of each of the Fibonacci n^{th} powers F_k^n is complete, and (as is obvious) that fewer than 2^{n-1} copies of $\{F_k^n\}$ does not yield a complete sequence.

Hoggatt and King [2] showed that the Fibonacci sequence with any term deleted is complete, but is no longer complete if any two terms are omitted. O'Connell [3] showed that the twofold sequence of Fibonacci squares mentioned above with any one of the first six terms deleted is still complete, but that the deletion of any term after the sixth or of any two terms will destroy completeness. In treating sequences of Fibonacci n^{th} powers, this led us to define a minimal sequence.

Definition. A minimal sequence is a complete sequence which is no longer complete if any element is deleted.

For each positive integer k, the sequence $\{F_i\}$ with F_k removed is a minimal sequence of Fibonacci numbers. Although only two minimal sequences 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ and 1, 1, 1, 1, 4, 9, 9, 25, 25, $\cdots$ can be obtained by deleting elements from the twofold sequence of Fibonacci squares, there are infinitely many minimal sequences comprised of Fibonacci squares. One can simply replace some term F_k^2 in a minimal sequence of Fibonacci squares by several terms F_i^2 whose sum is less than or equal to F_k^2 in a way which preserves minimality, as illustrated by the minimal sequences 1, 1, 1, 1, 1, 1, 1, 1, 9, 9, 25, 25, $\cdots$ (replacing 4 by 1, 1, 1, 1) or 1, 1, 1, 4, 4, 4, 9, 25, 25, $\cdots$ (replacing 9 by 4). For each n larger than 3, infinitely many minimal sequences can be obtained by deleting terms from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. However, one can obtain a particular minimal sequence from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers by deleting as many terms F_k^n as possible without destroying completeness before deleting any terms F_{k+1}^n, for $k = 2, 3, 4, \cdots$. We show in Sec. 4 that this yields the unique minimal sequence defined below.

Definition. A distinguished sequence of Fibonacci n^{th} powers is a sequence a which maps N onto $\{F_k^n \mid k \in N\}$ satisfying

(a) $$a_k \leq a_{k+1} \quad \text{for every} \quad k \in N,$$

(b) $$a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i \quad \text{for every} \quad k \in N,$$

(c) $$a_{k+1} \neq a_k \quad \text{implies} \quad 1 + \sum_{i=1}^{k-1} a_i < a_{k+1} \quad \text{for every} \quad k \in N.$$

In Theorem 4, we show that if

$$r = \left[\!\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]\!\right],$$

where $[\![\cdots]\!]$ is the greatest integer function, and $\{a_k\}$ is the distinguished sequence of Fibonacci n^{th} powers, then from some point on each F_i^n occurs exactly r or $r - 1$ times. Theorem 5 sharpens this result to show that for even n each F_i^n appears exactly r times from some point on.

2. BASIC IDENTITIES

The identities

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k \quad \text{and} \quad F_{k+1}F_{k+2} = F_kF_{k+3} + (-1)^k$$

follow easily by induction on k and show that

(1) $$\frac{F_{2k}}{F_{2k-1}} < \frac{F_{2k+1}}{F_{2k}}, \quad \frac{F_{2k}}{F_{2k-1}} < \frac{F_{2k+2}}{F_{2k+1}}, \quad \frac{F_{2k+1}}{F_{2k}} > \frac{F_{2k+3}}{F_{2k+2}}$$

for any positive integer k. Thus $\{F_{2k}/F_{2k-1}\}$ is an increasing sequence, $\{F_{2k+1}/F_{2k}\}$ is a decreasing sequence, and

(2) $$\frac{F_{2j}}{F_{2j-1}} < \frac{F_{2k+1}}{F_{2k}}$$

for all positive integers j and k. Since

$$\frac{F_{2k+1}}{F_{2k}} - \frac{F_{2k}}{F_{2k-1}} = \frac{1}{F_{2k}F_{2k-1}}$$

approaches zero as k approaches infinity, the sequences

$$\{F_{2k}, F_{2k-1}\}, \quad \{F_{2k+1}/F_{2k}\}, \quad \text{and} \quad \{F_{k+1}/F_k\}$$

have a common limit α. The identity

$$\frac{F_{k+2}}{F_{k+1}} = \frac{F_{k+1} + F_k}{F_{k+1}} = 1 + \frac{F_k}{F_{k+1}}$$

implies that $\alpha = 1 + 1/\alpha$, and α is clearly positive, so $\alpha = (1 + \sqrt{5})/2$ is the common limit. These are all well known properties of the Fibonacci numbers.

From (1) and (2), it is clear that $5/3 = F_5/F_4 \geq F_{k+1}/F_k$ except when $k = 2$. It follows by induction that

$$\left(\frac{F_{k+1}}{F_k}\right)^n \leq \left(\frac{5}{3}\right)^n < 1 + 2^{n-1} \quad (k \neq 2) \tag{3}$$

for all positive integers n and k with $k \neq 2$. Now the inequality

$$F_{k+1}^n \leq 1 + 2^{n-1}\sum_{i=1}^{k} F_i^n \quad (n, k \in N) \tag{4}$$

is true for $k = 1$ and $k = 2$, and for $k > 2$,

$$F_{k+1}^n < (1 + 2^{n-1})F_k^n = F_k^n + 2^{n-1}F_k^n \leq 1 + 2^{n-1}\sum_{i=1}^{k-1} F_i^n + 2^{n-1}F_k^n = 1 + 2^{n-1}\sum_{i=1}^{k} F_i^n$$

follows by induction on k.

3. COMPLETE SEQUENCES

It will frequently be helpful to use the following criterion due to Brown [1] in considering the completeness of various sequences.

<u>Completeness Criterion:</u> A non-decreasing sequence $\{a_i\}$ of positive integers is complete if and only if

$$a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i \tag{5}$$

for every non-negative integer k.

Brown's criterion and the inequality (4) are instrumental in proving the next theorem.

<u>Theorem 1.</u> For any positive integer n, the sequence of 2^{n-1} of each of the Fibonacci n^{th} powers is complete, but the sequence of $2^{n-1} - 1$ of each of the Fibonacci n^{th} powers is not complete.

Proof. Let $\{a_i\}$ be the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. That is,

$$a_k = F_m^n \quad \text{for} \quad (m-1)\cdot 2^{n-1} < k \le m \cdot 2^{n-1}.$$

If $k = 2^{n-1}\cdot m$ for some $m \in N$, then $a_{k+1} = F_{m+1}^n$ and

$$\sum_{i=1}^{k} a_i = 2^{n-1}\sum_{i=1}^{m} F_i^n ,$$

so the inequality (5) follows from (4). Otherwise, $a_{k+1} = a_k$ and the inequality (5) is clear. Hence $\{a_i\}$ is complete by the Completeness Criterion. $2^{n-1} - 1$ copies of $\{F_k^n\}$ is obviously not complete, since

$$F_3^n = 2^n > 1 + (2^{n-1} - 1)(F_1^n + F_2^n) = 2^n - 1 .$$

It is easy to see that the 2^{n-1}-fold sequence of Fibonacci n^{th} powers is not minimal, since in any case, one of the 2^n ones can be omitted. For $n \ge 4$, infinitely many terms can be omitted without destroying completeness, as is shown by the following theorem.

Theorem 2. Let n be a positive integer and let $r_2 = 2^n - 2$, $r_k = [\![(F_{k+1}/F_k)^n]\!]$ for each positive integer $k \ne 2$. Then the non-decreasing sequence of Fibonacci n^{th} powers in which, for each positive integer k, F_k^n occurs exactly r_k times, is complete.

Proof. The sequence $\{a_i\}$ defined in this theorem is given by taking $a_i = F_k^n$ when

$$\sum_{j=1}^{k-1} r_j < i \le \sum_{j=1}^{k} r_j .$$

The condition (5) is clear if $a_{k+1} = a_k$. Otherwise, we have $a_k = F_m^n$, $a_{k+1} = F_{m+1}^n$, and

$$\begin{aligned} a_{k+1} = F_{m+1}^n &\le (1 + r_m)F_m^n = F_m^n + r_m F_m^n \\ &\le 1 + \sum_{i=1}^{m-1} r_i F_i^n + r_m F_m^n = 1 + \sum_{i=1}^{m} r_i F_i^n . \end{aligned}$$

Therefore (5) follows by induction on m. The Completeness Criterion gives the theorem.

If $n \ge 4$ and $k \ge 3$,

$$r_k = [\![(F_{k+1}/F_k)^n]\!] \le [\![(5/3)^n]\!] \le 2^{n-1} - 1$$

by (3), and so at least one term F_k^n for each $k = 3, 4, 5, \cdots$ can be omitted from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. There is still no guarantee that the sequence given in Theorem 2 is minimal. (It is, if $n = 1$ or 2.) However, we will see in the next section that if n is even, then the sequence given in Theorem 2 is almost minimal, in the sense that a minimal sequence can be obtained from it by deleting a finite subsequence.

4. THE DISTINGUISHED SEQUENCES

We will now specialize our study to a particular minimal sequence of Fibonacci n^{th} powers. Let us recall our definition of a distinguished sequence.

Definition. A distinguished sequence of Fibonacci n^{th} powers is a sequence $\{a_k\}$ which maps N onto $\{F_k^n \mid k \in N\}$ satisfying

(a) $$a_k \leq a_{k+1} \quad \text{for every } k \in N\,,$$

(b) $$a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i \quad \text{for every } k \in N\,,$$

(c) $$a_{k+1} \neq a_k \quad \text{implies} \quad 1 + \sum_{i=1}^{k-1} a_i < a_{k+1} \quad \text{for every } k \in N\,.$$

We would like to show that for each positive integer n, a distinguished sequence exists and is unique. Starting with the complete 2^{n-1}-fold sequence of Fibonacci n^{th} powers, deleting one $F_2^n = 1$, and consecutively deleting enough F_j^n so that (b) and (c) are satisfied for $j = 3, 4, 5, \cdots$, it is clear that one can construct a sequence satisfying (a), (b), and (c). The inequalities $F_{j-1}^n + F_j^n \leq (F_{j-1} + F_j)^n = F_{j+1}^n$, (b), and (c) insure that $\{a_k\}$ is onto $\{F_i^n \mid i \in N\}$. Properties (b) and (c) also guarantee uniqueness. Henceforth, we will call the unique distinguished sequence of Fibonacci n^{th} powers the n^{th} distinguished sequence.

The work [2] of Hoggatt and King shows that the first distinguished sequence $\{a_i\}$ is the sequence 1, 2, 3, 5, 8, 13, $\cdots$ defined by taking $a_i = F_{i+1}$, and O'Connells' work [3] shows that the second distinguished sequence $\{a_i\}$ is the sequence 1, 1, 1, 4, 4, 9, 9, 25, 25, 64, 64, $\cdots$ defined by taking $a_i = F_j^2$ for $j = [\![i/2]\!] + 1$.

Theorems 4 and 5 give information about the n^{th} distinguished sequence for any positive integer n. Before stating these theorems we will introduce some notation and recall some well known facts concerning Fibonacci and Lucas numbers.

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ be the roots of the equation $x^2 = x + 1$, and note that $\alpha\beta = -1$. Recall that

$$\alpha = \lim_{k \to \infty} (F_{k+1}/F_k)\,.$$

Multiplying $x^2 = x + 1$ by x^n, we see that

$$\begin{aligned} \alpha^{n+2} &= \alpha^{n+1} + \alpha^n\,, \\ \beta^{n+2} &= \beta^{n+1} + \beta^n\,, \\ (\alpha^{n+2} + \beta^{n+2}) &= (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n)\,. \end{aligned} \tag{6}$$

Defining

(7) $$L_n = \alpha^n + \beta^n = \alpha^n + (-1/\alpha)^n$$

for $n = 0, 1, 2, 3, \cdots$, we see that $L_0 = 2$, $L_1 = 1$ (since $\alpha = 1 + 1/\alpha$), and $L_{n+1} = L_n + L_{n-1}$ for any positive integer n. The integers L_n defined in this way are the usual Lucas numbers. Now $\alpha > 1$, so $0 < 1/\alpha^n < 1$ for each positive integer n, and it follows from (7) that

(8) $$[\![\alpha^n]\!] = \alpha^n - \frac{1}{\alpha^n} = L_n \quad (n \text{ odd}),$$

$$[\![\alpha^n]\!] = \alpha^n + \frac{1}{\alpha^n} - 1 = L_n - 1 \quad (n \text{ even}),$$

and thence

(9) $$[\![\alpha^n]\!] < \alpha^n < [\![\alpha^n]\!] + 1 \quad (n \in N).$$

Also, $\alpha^2 = \alpha + 1 > 2$, so $0 < 2/\alpha^n < 1$ for any even positive integer n, and

(10) $$1/\alpha^n + [\![\alpha^n]\!] < \alpha^n < [\![\alpha^n]\!] + 1 \quad (n \text{ even})$$

follows from (8).

Lemma 3. For each positive integer n there is a positive integer M such that, for $k \geq M$,

$$[\![\alpha^n]\!] < (F_{k+1}/F_k)^n < [\![\alpha^n]\!] + 1,$$

and, if n is even,

$$(F_{k-1}/F_k)^n + [\![\alpha^n]\!] < (F_{k+1}/F_k)^n < [\![\alpha^n]\!] + 1.$$

Proof. This lemma is immediate from (9), (10), and the limits,

$$\lim_{k\to\infty} (F_{k+1}/F_k)^n = \alpha^n, \qquad \lim_{k\to\infty} (F_{k-1}/F_k)^n = 1/\alpha^n.$$

We are now ready to prove Theorems 4 and 5.

Theorem 4. Let $\{a_i\}$ be the n^{th} distinguished sequence, and let $r = [\![\alpha^n]\!]$, where $\alpha = (1 + \sqrt{5})/2$. Then there is a positive integer M such that for each $k \geq M$, $a_i = F_k^n$ for exactly r or $r - 1$ values of i.

Proof. Let M be as in Lemma 3. It suffices to show that if $k \geq M$ and

$$h = \min\{i \mid a_i = F_k^n\},$$

then

$$1 + \sum_{i=1}^{h-1} a_i + (r - 2)F_k^n < F_{k+1}^n \leq 1 + \sum_{i=1}^{h-1} a_i + rF_k^n.$$

We know by property (c) of the distinguished sequence that

$$1 + \sum_{i=1}^{h-2} a_i = 1 + \sum_{i=1}^{h-1} a_i - F_{k-1}^n < F_k^n ,$$

and so

$$1 + \sum_{i=1}^{h-1} a_i + (r - 2)F_k^n < F_{k-1}^n + (r - 1)F_k^n < rF_k^n < F_{k+1}^n$$

since

$$r < (F_{k+1}/F_k)^n .$$

Also,

$$F_k^n \le 1 + \sum_{i=1}^{h-1} a_i$$

by property (b) of the distinguished sequence, so

$$F_{k+1}^n < (r + 1)F_k^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n ,$$

since

$$(F_{k+1}/F_k)^n < r + 1 .$$

We can sharpen this result for even values of n.

Theorem 5. Let n be an even positive integer, let $\{a_i\}$ be the n^{th} distinguished sequence, and let $r = [\![\alpha^n]\!]$ for $\alpha = (1 + \sqrt{5})/2$. Then there is a positive integer M such that for each $k \ge M$, $a_i = F_k^n$ for exactly r values of i.

Proof. Let M be as in Lemma 3. It suffices to show that if $k \ge M$ and

$$h = \min\{i \mid a_i = F_k^n\} ,$$

then

$$1 + \sum_{i=1}^{h-1} a_i + (r - 1)F_k^n < F_{k+1}^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n .$$

By property (c) of the distinguished sequence $\{a_i\}$,

$$1 + \sum_{i=1}^{h-2} a_i = 1 + \sum_{i=1}^{h-1} a_i - F_{k-1}^n < F_k^n$$

so

$$1 + \sum_{i=1}^{h-1} a_i + (r - 1)F_k^n < F_{k-1}^n + rF_k^n < F_{k+1}^n ,$$

since

$$(F_{k-1}/F_k)^n + r < (F_{k+1}/F_k)^n .$$

By property (b),

$$F_k^n \le 1 + \sum_{i=1}^{h-1} a_i, \quad \text{so} \quad F_{k+1}^n < (r+1)F_k^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n ,$$

since

$$(F_{k+1}/F_k)^n < r + 1 .$$

Now we define a complete sequence to be <u>almost minimal</u> if it can be made minimal by deleting a finite subsequence.

<u>Corollary 6.</u> If n is even, the sequence defined in Theorem 2 is almost minimal.

<u>Proof</u>. The sequence $\{a_i\}$ defined in Theorem 2 is given by taking

$$r_k = [\![(F_{k+1}/F_k)^n]\!]$$

for each positive integer $k \ne 2$, $r_2 = 2^n - 2$, and taking $a_i = F_k^n$ when

$$\sum_{j=1}^{k-1} r_j < i \le \sum_{j=1}^{k} r_j .$$

Since $r_k \ge 1$ for every $k \ne 2$, the sequence $\{a_i\}$ is onto the set $\{F_k^n \mid k \in N\}$. Since $\{a_i\}$ is a complete sequence of non-decreasing terms, it satisfies properties (a) and (b) of the n^{th} distinguished sequence. For M as in Lemma 3, $r_k = [\![\alpha^n]\!] = r$ when $k \ge M$. Thus, the minimal n^{th} distinguished sequence can be obtained from the sequence $\{a_i\}$ by deleting finitely many terms F_k^n with $k \le M$.

When n is even, Corollary 6 provides a fairly efficient means of constructing the n^{th} distinguished sequence in a finite number of steps. First M is determined by inspection, for example, and then enough of the r_k terms F_k^n are deleted to make the sequence satisfy property (c) of the distinguished sequence, for $k = 3, 4, \cdots, M - 1$. For example, when $n = 4$, we have $M = 5$ and $r_1 = 1$, $r_2 = 14$, $r_3 = 5$, $r_4 = 7$, $r_k = 6$ $(k \ge 5)$. So the sequence of Theorem 2 has fifteen 1's, five 16's, seven 81's, and six F_k^4's for all $k \ge 5$. Since none of the F_k^4 can be deleted for $k < 5$, this sequence is already the fourth distinguished sequence.

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," <u>Amer. Math. Monthly</u>, Vol. 68 (1961), pp. 557-560.
2. V. E. Hoggatt, Jr., and C. King, Problem E1424, <u>Amer. Math. Monthly</u>, Vol. 67 (1960), p. 593.
3. Roger O'Connell, "Representations of Integers as Sums of Fibonacci Squares," <u>Fibonacci Quarterly</u>, 10 (1972), pp. 103-111.