ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-227 Proporsed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj+ck)^m(bj+dk)^n$$

$$= m!n!\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n}{r}a^{m-r}d^{n-r}(bc)^r .$$

In particular, show that the Legendre polynomial $P_n(x)$ satisfies

$$(n!)^2P_n(x) = \sum_{j,k=0}^{n}(-1)^{j+k}\binom{n}{j}\binom{n}{k}(aj+ck)^n(bj+dk)^n ,$$

where

$$ad = \frac{1}{2}(x+1), \qquad bc = \frac{1}{2}(x-1) .$$

H-228 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the sequence $\{u_n\}_{n=1}^{\infty}$ as follows: $u_n = (F_n)^{F_n}$ $(n \geq 1)$, where F_n denotes the n^{th} Fibonacci number.

(1). Find a recurrence relation for $\{u_n\}_{n=1}^{\infty}$ and
(2). Find a generating function for the sequence, $\{u_n\}_{n=1}^{\infty}$.

H-229 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array

$$A(n,k) \qquad (0 \le k \le n)$$

is defined by means of

$$(*) \qquad \begin{cases} A(n+1,\ 2k) = A(n,\ 2k-1) + aA(n,\ 2k) \\ A(n+1,\ 2k+1) = A(n,\ 2k) + bA(n,\ 2k+1) \end{cases}$$

together with

$$A(0,0) = 1, \quad A(0,k) = 0 \quad (k \ne 0) .$$

Find $A(n,k)$ and show that

$$\sum_k A(n,\ 2k)(ab)^k = a(a+b)^{n-1}, \qquad \sum_k A(n,\ 2k+1)(ab)^k = (a+b)^{n-1} .$$

SOLUTIONS

ARRAY OF HOPE

H-195 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California

Consider the array indicated below:

1	1						
1	2						
2	4	1	1				
5	9	3	4				
13	22	7	11	1	1		
34	56	16	27	5	6		
89	145	38	65	16	22	1	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

(i) Show that the row sums are F_{2n}, $n \ge 2$.
(ii) Show that the rising diagonal sums are the convolution of

$$\{F_{2n-1}\}_{n=0}^{\infty} \quad \text{and} \quad \{u(n;\ 2,2)\}_{n=0}^{\infty} ,$$

the generalized numbers of Harris and Styles.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let $A(n,k)$ denote the element in the n^{th} row and k^{th} column. Then (presumably)

$$A(n,1) = F_{2n-3} \qquad (n > 1)$$

and

$$\begin{cases} A(n,\ 2k) = A(n,\ 2k-1) + A(n-1,\ 2k) \\ A(n,\ 2k+1) = A(n-1,\ 2k+1) + A(n-2,\ 2k) \end{cases} \qquad (k \geq 1)\ .$$

Put

$$F(x,y) = \sum_{n=1}^{\infty} \sum_{k} A(n,\ 2k)\, x^n y^{2k}$$

$$G(x,y) = \sum_{n=1}^{\infty} \sum_{k} A(n,\ 2k+1)\, x^n y^{2k+1}$$

$$A(x) = \sum_{n=1}^{\infty} A(n,1)\, x^n\ .$$

Then

$$A(x) = x + \sum_{n=2}^{\infty} F_{2n-3} x^n = x + x \sum_{n=1}^{\infty} F_{2n-1} x^n$$

$$= x + x\,\frac{x - x^2}{1 - 3x + x^2} = \frac{x - 2x^2}{1 - 3x + x^2}\ .$$

Next

$$F(x,y) = \sum_{n} \sum_{k} (A(n,\ 2k-1) + A(n-1,\ 2k))\, x^n y^{2k}$$

$$= xF(x,y) + yG(x,y)\ ,$$

so that

$$(1 - x)F(x,y) = yG(x,y)\ . \tag{1}$$

Also,

$$G(x,y) = y\sum_{1}^{\infty} A(n,1)x^n + \sum_{n=2}^{\infty}\sum_{k>0} (A(n-1,\ 2k+1) + A(n-2,\ 2k))\,x^n y^{2k+1}$$

$$= yA(x) + x\sum_{n=1}^{\infty}\sum_{k>0} A(n,\ 2k+1)x^n y^k + x^2y\sum_{n=1}^{\infty}\sum_{k>0} A(n,\ 2k)x^n y^{2k}$$

$$= y(1-x)A(x) + xG(x,y) + x^2yF(x,y)\ ,$$

so that

$$(2)\qquad (1-x)G(x,y) = x^2yF(x,y) + \frac{x(1-x)(1-2x)y}{1-3x+x^2}\ .$$

It follows from (1) and (2) that

$$(3)\qquad \begin{cases} ((1-x)^2 - x^2y^2)F(x,y) = \dfrac{x(1-x)(1-2x)y^2}{1-3x+x^2} \\[2ex] ((1-x)^2 - x^2y^2)G(x,y) = \dfrac{x(1-x)^2(1-2x)y}{1-3x+x^2} \end{cases}\ .$$

Hence

$$(4)\qquad ((1-x)^2 - x^2y^2)\sum_{n}\sum_{k} A(n,k)x^n y^k = \frac{xy(1-x)(1-2x)(1-x+y)}{1-3x+x^2}\ .$$

For $y = 1$ this reduces to

$$\sum_{n} x^n \sum_{k} A(n,k) = \frac{x(1-x)(2-x)}{1-3x+x^2}$$

$$= x + \frac{x}{1-3x+x^2}$$

$$= x + \sum_{1}^{\infty} F_{2n}x^n\ ,$$

so that

$$\sum_{k} A(n,k) = F_{2n} \qquad (n > 1)\ .$$

If we take $y = x$, Eq. (4) reduces to

$$\sum_{n=2}^{\infty} x^n \sum_k A(n-k,\ k) = \frac{x^2(1-x)(1-2x)}{(1-x-x^2)(1-x+x^2)(1-3x+x^2)}$$

$$= \sum_{1}^{\infty} F_{2n-1}x^n \; \frac{x(1-2x)}{(1-x-x^2)(1-x+x^2)} \quad .$$

This expresses the rising diagonal sums

$$\sum_{k=1}^{n-1} A(n-k,\ k)$$

as convolutions as stated.

Remark. It follows from (3) that

$$\begin{cases} F(x,y) = \dfrac{1-2x}{1-3x+x^2} \displaystyle\sum_{k=1}^{\infty} \frac{x^{2k-1}y^{2k}}{(1-x)^{2k-1}} \\[2ex] G(x,y) = \dfrac{1-2x}{1-3x+x^2} \displaystyle\sum_{k=1}^{\infty} \frac{x^{2k-1}y^{2k-1}}{(1-x)^{2k-2}} \end{cases} ,$$

so that

$$(5) \qquad \begin{cases} \displaystyle\sum_{n=2k-1}^{\infty} A(n,2k)x^n = \frac{(1-2x)x^{2k-1}}{(1-3x+x^2)(1-x)^{2k-1}} \\[2ex] \displaystyle\sum_{n=2k-1}^{\infty} A(n,\ 2k-1)x^n = \frac{(1-2x)x^{2k-1}}{(1-3x+x^2)(1-x)^{2k-2}} \end{cases} .$$

By means of (5) we can obtain explicit formulas for $A(n,k)$. Since

$$\frac{1-2x}{1-3x+x^2} = \sum_{r=0}^{\infty} F_{2r-1}x^r ,$$

it follows that

$$\sum_{n=2k-1}^{\infty} A(n,2k)x^n = x^{2k-1} \sum_{r=0}^{\infty} F_{2r-1}x^r \sum_{s=0}^{\infty} \binom{2k+s-2}{s} x^s .$$

Therefore,

$$A(n,2k) = \sum_{r=0}^{n-2k+1} \binom{n-r-1}{2k-2} F_{2r-1} \quad .$$

Similarly

$$A(n,\ 2k-1) = \sum_{r=0}^{n-2k+1} \binom{n-r-2}{2k-3} F_{2r-1} \qquad (k > 1)\ .$$

Also solved by the Proposer.

PARTITION

H-196 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

(a) Let A_0 be the set of integral parts of the positive integral multiples of τ, where

$$\tau = \frac{1+\sqrt{5}}{2} \quad ,$$

and let A_{m+1}, $m = 0, 1, 2, \cdots$, be the set of integral parts of the numbers $n\tau^2$ for $n \in A_m$. Prove that the collection of Z^+ of all positive integers is the disjoint union of the A_j.

(b) Generalize the proposition in (a).

Solution by L. Carlitz, Duke University, Durham, North Carolina.

1. Put

$$a(n) = [n\tau]\ , \qquad b(n) = [n\tau^2] = [n(\tau+1)] = a(n) + n\ .$$

Also for brevity put

$$(a) = \{a(n)\,|\,n = 1, 2, 3, \cdots\}\ ,$$
$$(b) = \{b(n)\,|\,n = 1, 2, 3, \cdots\}\ .$$

It is well known that*

$$Z^+ = (a) \cup (b)\ . \qquad (*)$$

Put

$$(b^k a) = \{b^k a(n)\,|\,n = 1, 2, 3, \cdots\}\ ,$$

where juxtaposition denotes composition. Then it follows at once from (*) that

$$\begin{aligned} Z^+ &= (a) \cup (ba) \cup (b^2) \\ &= (a) \cup (ba) \cup (b^2a) \cup (b^3) , \end{aligned}$$

and so on. Clearly this implies

$$Z^+ = \bigcup_{k=0}^{\infty} (b^k a) = \bigcup_{k=0}^{\infty} A_k .$$

2. Let α, β be positive irrational numbers such that

$$(1/\alpha) + (1/\beta) = 1$$

and put

$$a(n) = [\alpha n] , \qquad b(n) = [\beta n] .$$

Then it is well known that

$$Z^+ = (a) \cup (b) ,$$

where, as above,

$$(a) = \{a(n) \mid n = 1, 2, 3, \cdots\} , \qquad b(n) = \{b(n) \mid n = 1, 2, 3, \cdots\} .$$

Hence

$$\begin{aligned} Z^+ &= (a) \cup (ba) \cup (b^2) \\ &= (a) \cup (ba) \cup (b^2a) \cup (b^3) , \end{aligned}$$

and so on. Thus

$$Z^+ = \bigcup_{k=0}^{\infty} (b^k a) .$$

Remark. The functions a(n), b(n) in 1 are studied in considerable detail in the paper by L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville: "Fibonacci Representations," Fibonacci Quarterly, Vol. 10, No. 1, pp. 1-28.

Also solved by the Proposer.

Editorial Note: See Beatty's Theorem (American Math. Monthly, 33 (1926), 159, and 34 (1927) 159.)

The editor wishes to acknowledge solutions to H-194 by L. Frohman, P. Bruckman, and J. Ivie.

Editorial Note: The following list represents previous problem proposals (less than or equal to H-100) which, to date, have not been solved: 22, 23, 40, 43, 46, 60, 61, 73, 76, 77, 84, 87, 90, 91, 94, and 100. Starting in the next section, we shall re-run some of these proposals.

ERRATA

On Problem H-218, April, 1973, please change the matrix to read:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & & \\ 0 & 1 & 1 & 0 & \cdots & \\ \cdots & 0 & 2 & \cdots & & \end{pmatrix}_{n \times n}$$