# A SOLUTION TO THE CLASSICAL PROBLEM OF FINDING SYSTEMS OF THREE MUTUALLY ORTHOGONAL NUMBERS IN A CUBE FORMED BY THREE SUPERIMPOSED 10 X 10 X 10 CUBES

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## INTRODUCTION

In 1779, Euler conjectured that no pair of orthogonal squares exist for $n \equiv 2 \pmod 4$. Then in 1959, the Euler conjecture was shown to be incorrect by Bose, Shrikande and Parker [1]. Recently (in 1972), Hoggatt and this author extended Bose, Shrikande and Parker's work by finding a way to make the $10 \times 10$ square pairwise orthogonal as well as magic (for a square to be magic, each of the two diagonals must have the same sum as in every row and in every column) [2]. The work done on this difficult problem was then extended by this author, who found a solution to the classical Eulerian magic cube problem of order ten [3]. Then this author was fortunate enough to receive some letters from the great mathematician, Professor Erdös. Professor Erdös introduced me to one of the most difficult and unsolved problems of our time: namely, the 200-year-old question of whether it is possible to find systems of three mutually orthogonal numbers in a cube of three superimposed $10 \times 10 \times 10$ Latin cubes.

## ABSTRACT

In this paper, we have succeeded in constructing for the first time certain systems of three mutually orthogonal numbers in a cube of three superimposed $10 \times 10 \times 10$ Latin cubes (the letters used are A, B, C, $\cdots$, J).

In our construction (Tables 1 through 10), we find ten separate groups (where each group consists of 100 cells and each cell contains three letters) such that each and every cell in a single group (we consider one group at a time) is in a different file, different column, and different row; and also (this is the major requirement) in any one group when we compare each and every one of the 100 cells to one another, the three letters in each and every cell in the group are mutually (three pairwise) orthogonal. In the construction of our cubes in Tables 1-10, we find in each cell three capital letters of the alphabet followed by a comma and then a digit (the digits range through 0, 1, 2, $\cdots$, 9). The digits on the right denote the group to which the three letters in the cell belong. For example: the three letters in each of the 100 cells throughout the cube that end in ,0 denote a single group (say) G(,0) and in this group G(,0) when we compare each and every one of the 100 cells to one another, the three letters in each and every cell in the group G(,0) are mutually (three pairwise) orthogonal. In the exact way we found the orthogonal properties of group G(,0) we find the identical orthogonal properties in the remaining nine groups G(,1), G(,2), $\cdots$, G(,9).

In our construction, it is also possible to find three pairwise orthogonal letters in a system of files where each file is in a different row, and different column (we use our top $10 \times 10$ square (square number 0, Table 1) as a reference for the coordinates of our rows and columns). An example of a single file (all files are considered to begin on square number 0, abbreviated SN0) is the ten cells in f(HGD, 0) = (HGD, 0) on square number 0 + (IHC, 1) on SN1 + (FJA, 2) on SN2 + ⋯ + (BFI, 9) on SN9. Then we define a group of files (say F(, 0)) ending in , 0 as the 100 cells in F(, 0) = f(HGD, 0) + f(HCI, 0) + ⋯ + f(HJH, 0). Now in F(, 0) when we compare each and every one of the cells (100 cells) to one another, the three letters in each and every cell in F(, 0) are mutually (three pairwise) orthogonal. In the exact way we found the orthogonal properties of F(, 0), we find the identical orthogonal properties in the remaining F(, 1), F(, 2), ⋯, F(, 9).

Remark. Using the exact methods that were used to construct the cubes in Tables 1-10, this author has extended the remarkable results on singly-even orthogonal squares by Bose, Shrikande and Parker [1], since we have generalized the construction technique and are able to find systems (exactly like the systems in this paper) of three pairwise orthogonal numbers in all (except $2^3$ and $6^3$) cubes formed by three superimposed Latin cubes. It is also possible to show: if a construction for a square $(2P)^2$ is known we are then always able to construct a cube $(2P(2m + 1))^3$ with the exact three pairwise orthogonal properties we have shown in this paper ($P > 3$ is an odd prime and $m = 0, 1, \cdots$). However, since this author has not resolved (to his satisfaction) the question: "Is it possible to superimpose three mutually orthogonal $10 \times 10$ Latin squares?," we shall discuss our methods in a future paper.

Table 1

Square Number 0

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| HGD,0 | GIG,1 | DEA,5 | CHJ,3 | BBF,2 | ICI,9 | EAB,4 | FFE,7 | ADC,6 | JJH,8 |
| AHJ,6 | CGD,3 | BJH,2 | EFE,4 | GDC,1 | FAB,7 | HCI,0 | JIG,8 | IEA,9 | DBF,5 |
| GJH,1 | FEA,7 | IGD,9 | AAB,6 | DHJ,5 | EDC,4 | JFE,8 | CBF,3 | BCI,2 | HIG,0 |
| JEA,8 | IFE,9 | CAB,3 | BGD,2 | ECI,4 | HBF,0 | GHJ,1 | AJH,6 | DIG,5 | FDC,7 |
| CDC,3 | ACI,6 | HFE,0 | GBF,1 | JGD,8 | BIG,2 | DJH,5 | IAB,9 | FHJ,7 | EEA,4 |
| FCI,7 | EJH,4 | JDC,8 | HEA,0 | IIG,9 | DGD,5 | ABF,6 | BHJ,2 | CFE,3 | GAB,1 |
| DFE,5 | BDC,2 | AIG,6 | JCI,8 | HAB,0 | CJH,3 | FGD,7 | GEA,1 | EBF,4 | IHJ,9 |
| IBF,9 | HHJ,0 | GCI,1 | DDC,5 | FJH,7 | AEA,6 | CIG,3 | EGD,4 | JAB,8 | BFE,2 |
| EIG,4 | DAB,5 | FBF,7 | IJH,9 | AFE,6 | JHJ,8 | BEA,2 | HDC,0 | GGD,1 | CCI,3 |
| BAB,2 | JBF,8 | EHJ,4 | FIG,7 | CEA,3 | GFE,1 | IDC,9 | DCI,5 | HJH,0 | AGD,6 |

Table 2

Square Number 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| IHC,1 | FDI,6 | GGE,9 | JIF,0 | EJD,7 | DBG,2 | HEH,3 | ACB,8 | BFJ,4 | CAA,5 |
| BIF,4 | JHC,0 | EAA,7 | HCB,3 | FFJ,6 | AEH,8 | IBG,1 | CDI,5 | DGE,2 | GJD,9 |
| FAA,6 | AGE,8 | DHC,2 | BEH,4 | GIF,9 | HFJ,3 | CCB,5 | JJD,0 | EBG,7 | IDI,1 |
| CGE,5 | DCB,2 | JEH,0 | EHC,7 | HBG,3 | IJD,1 | FIF,6 | BAA,4 | GDI,9 | AFJ,8 |
| JFJ,0 | BBG,4 | ICB,1 | FJD,6 | CHC,5 | EDI,7 | GAA,9 | DEH,2 | AIF,8 | HGE,3 |
| ABG,8 | HAA,3 | CFJ,5 | IGE,1 | DDI,2 | GHC,9 | BJD,4 | EIF,7 | JCB,0 | FEH,6 |
| GCB,9 | EFJ,7 | BDI,4 | CBG,5 | IEH,1 | JAA,0 | AHC,8 | FGE,6 | HJD,3 | DIF,2 |
| DJD,2 | IIF,1 | FBG,6 | GFJ,9 | AAA,8 | BGE,4 | JDI,0 | HHC,3 | CEH,5 | ECB,7 |
| HDI,3 | GEH,9 | AJD,8 | DAA,2 | BCB,4 | CIF,5 | EGE,7 | IFJ,1 | FHC,6 | JBG,0 |
| EEH,7 | CJD,5 | HIF,3 | ADI,8 | JGE,0 | FCB,6 | DFJ,2 | GBG,9 | IAA,1 | BHC,4 |

Table 3

Square Number 2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| FJA,2 | DCE,4 | BHB,8 | ABD,9 | JAI,6 | EGF,0 | IIJ,5 | CDH,1 | GEG,3 | HFC,7 |
| GBD,3 | AJA,9 | JFC,6 | IDH,5 | DEG,4 | CIJ,1 | FGF,2 | HCE,7 | EHB,0 | BAI,8 |
| DFC,4 | CHB,1 | EJA,0 | GIJ,3 | BBD,8 | IEG,5 | HDH,7 | AAI,9 | JGF,6 | FCE,2 |
| HHB,7 | EDH,0 | AIJ,9 | JJA,6 | IGF,5 | FAI,2 | DBD,4 | GFC,3 | BCE,8 | CEG,1 |
| AEG,9 | GGF,3 | FDH,2 | DAI,4 | HJA,7 | JCE,6 | BFC,8 | EIJ,0 | CBD,1 | IHB,5 |
| CGF,1 | IFC,5 | HEG,7 | FHB,2 | ECE,0 | BJA,8 | GAI,3 | JBD,6 | ADH,9 | DIJ,4 |
| BDH,8 | JEG,6 | GCE,3 | HGF,7 | FIJ,2 | AFC,9 | CJA,1 | DHB,4 | IAI,5 | EBD,0 |
| EAI,0 | FBD,2 | DGF,4 | BEG,8 | CFC,1 | GHB,3 | ACE,9 | IJA,5 | HIJ,7 | JDH,6 |
| ICE,5 | BIJ,8 | CAI,1 | EFC,0 | GDH,3 | HBD,7 | JHB,6 | FEG,2 | DJA,4 | AGF,9 |
| JIJ,6 | HAI,7 | IBD,5 | CCE,1 | AHB,9 | DDH,4 | EEG,0 | BGF,8 | FFC,2 | GJA,3 |

Table 4

Square Number 3

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| JIH,3 | EFA,9 | HCD,2 | GDB,7 | AEJ,0 | CAC,6 | FJE,8 | BHF,5 | DBI,1 | IGG,4 |
| DDB,1 | GIH,7 | AGG,0 | FHF,8 | EBI,9 | BJE,5 | JAC,3 | IFA,4 | CCD,6 | HEJ,2 |
| EGG,9 | BCD,5 | CIH,6 | DJE,1 | HDB,2 | FBI,8 | IHF,4 | GEJ,7 | AAC,0 | JFA,3 |
| ICD,4 | CHF,6 | GJE,7 | AIH,0 | FAC,8 | JEJ,3 | EDB,9 | DGG,1 | HFA,2 | BBI,5 |
| GBI,7 | DAC,1 | JHF,3 | EEJ,9 | IIH,4 | AFA,0 | HGG,2 | CJE,6 | BDB,5 | FCD,8 |
| BAC,5 | FGG,8 | IBI,4 | JCD,3 | CFA,6 | HIH,2 | DEJ,1 | ADB,0 | GHF,7 | EJE,9 |
| HHF,2 | ABI,0 | DFA,1 | IAC,4 | JJE,3 | GGG,7 | BIH,5 | ECD,9 | FEJ,8 | CDB,6 |
| CEJ,6 | JDB,3 | EAC,9 | HBI,2 | BGG,5 | DCD,1 | GFA,7 | FIH,8 | IJE,4 | AHF,0 |
| FFA,8 | HJE,2 | BEJ,5 | CGG,6 | DHF,1 | IDB,4 | ACD,0 | JBI,3 | EIH,9 | GAC,7 |
| AJE,0 | IEJ,4 | FDB,8 | BFA,5 | GCD,7 | EHF,9 | CBI,6 | HAC,2 | JGG,3 | DIH,1 |

Table 5

Square Number 4

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| AAG,4 | CEB,7 | EFF,1 | ICH,8 | GDE,5 | FJJ,3 | DBA,9 | HGC,6 | JID,2 | BHI,0 |
| JCH,2 | IAG,8 | GHI,5 | DGC,9 | CID,7 | HBA,6 | AJJ,4 | BEB,0 | FFF,3 | EDE,1 |
| CHI,7 | HFF,6 | FAG,3 | JBA,2 | ECH,1 | DID,9 | BGC,0 | IDE,8 | GJJ,5 | AEB,4 |
| BFF,0 | FGC,3 | IBA,8 | GAG,5 | DJJ,9 | ADE,4 | CCH,7 | JHI,2 | EEB,1 | HID,6 |
| IID,8 | JJJ,2 | AGC,4 | CDE,7 | BAG,0 | GEB,5 | EHI,1 | FBA,3 | HCH,6 | DFF,9 |
| HJJ,6 | DHI,9 | BID,0 | AFF,4 | FEB,3 | EAG,1 | JDE,2 | GCH,5 | IGC,8 | CBA,7 |
| EGC,1 | GID,5 | JEB,2 | BJJ,0 | ABA,4 | IHI,8 | HAG,6 | CFF,7 | DDE,9 | FCH,3 |
| FDE,3 | ACH,4 | CJJ,7 | EID,1 | HHI,6 | JFF,2 | IEB,8 | DAG,9 | BBA,0 | GGC,5 |
| DEB,9 | EBA,1 | HDE,6 | FHI,3 | JGC,2 | BCH,0 | GFF,5 | AID,4 | CAG,7 | IJJ,8 |
| GBA,5 | BDE,0 | DCH,9 | HEB,6 | IFF,8 | CGC,7 | FID,3 | EJJ,1 | AHI,4 | JAG,2 |

Table 6

Square Number 5

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ECF,5 | BGJ,3 | FII,0 | DEC,6 | HFH,4 | GDA,8 | CHG,2 | JAD,9 | IJB,7 | ABE,1 |
| IEC,7 | DCF,6 | HBE,4 | CAD,2 | BJB,3 | JHG,9 | EDA,5 | AGJ,1 | GII,8 | FFH,0 |
| BBE,3 | JII,9 | GCF,8 | IHG,7 | FEC,0 | CJB,2 | AAD,1 | DFH,6 | HDA,4 | EGJ,5 |
| AII,1 | GAD,8 | DHG,6 | HCF,4 | CDA,2 | EFH,5 | BEC,3 | IBE,7 | FGJ,0 | JJB,9 |
| DJB,6 | IDA,7 | EAD,5 | BFH,3 | ACF,1 | HGJ,4 | FBE,0 | GHG,8 | JEC,9 | CII,2 |
| JDA,9 | CBE,2 | AJB,1 | EII,5 | EGJ,8 | FCF,0 | IFH,7 | HEC,4 | DAD,6 | BHG,3 |
| FAD,0 | HJB,4 | IGJ,7 | ADA,1 | EHG,5 | DBE,6 | JCF,9 | BII,3 | CFH,2 | GEC,8 |
| GFH,8 | EEC,5 | BDA,3 | FJB,0 | JBE,9 | III,7 | DGJ,6 | CCF,2 | AHG,1 | HAD,4 |
| CGJ,2 | FHG,0 | JFH,9 | GBE,8 | IAD,7 | AEC,1 | HII,4 | EJB,5 | BCF,3 | DDA,6 |
| HHG,4 | AFH,1 | CEC,2 | JGJ,9 | DII,6 | BAD,3 | GJB,8 | FDA,0 | EBE,5 | ICF,7 |

Table 7

Square Number 6

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| GEE,6 | ABC,2 | CDG,4 | HJI,5 | IGB,3 | BHH,7 | JFD,1 | DIA,0 | EAF,8 | FCJ,9 |
| EJI,8 | HEE,5 | ICJ,3 | JIA,1 | AAF,2 | DFD,0 | GHH,6 | FBC,9 | BDG,7 | CGB,4 |
| ACJ,2 | DDG,0 | BEE,7 | EFD,8 | CJI,4 | JAF,1 | FIA,9 | HGB,5 | IHH,3 | GBC,6 |
| FDG,9 | BIA,7 | HFD,5 | IEE,3 | JHH,1 | GGB,6 | AJI,2 | ECJ,8 | CBC,4 | DAF,0 |
| HAF,5 | EHH,8 | GIA,6 | AGB,2 | FEE,9 | IBC,3 | CCJ,4 | BFD,7 | DJI,0 | JDG,1 |
| DHH,0 | JCJ,1 | FAF,9 | GDG,6 | BBC,7 | CEE,4 | EGB,8 | IJI,3 | HIA,5 | AFD,2 |
| CIA,4 | IAF,3 | EBC,8 | FHH,9 | GFD,6 | HCJ,5 | DEE,0 | ADG,2 | JGB,1 | BJI,7 |
| BGB,7 | GJI,6 | AHH,2 | CAF,4 | DCJ,0 | EDG,8 | HBC,5 | JEE,1 | FFD,9 | IIA,3 |
| JBC,1 | CFD,4 | DGB,0 | BCJ,7 | EIA,8 | FJI,9 | IDG,3 | GAF,6 | AEE,2 | HHH,5 |
| IFD,3 | FGB,9 | JJI,1 | DBC,0 | HDG,5 | AIA,2 | BAF,7 | CHH,4 | GCJ,6 | EEE,8 |

Table 8

Square Number 7

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| DDJ,7 | HHD,8 | ABH,3 | BAE,1 | CIC,9 | JFB,4 | GGI,0 | IJG,2 | FCA,5 | EEF,6 |
| FAE,5 | BDJ,1 | CEF,9 | GJG,0 | HCA,8 | IGI,2 | DFB,7 | EHD,6 | JBH,4 | AIC,3 |
| HEF,8 | IBH,2 | JDJ,4 | FGI,5 | AAE,3 | GCA,0 | EJG,6 | BIC,1 | CFB,9 | DHD,7 |
| EBH,6 | JJG,4 | BGI,1 | CDJ,9 | GFB,0 | DIC,7 | HAE,8 | FEF,5 | AHD,3 | ICA,2 |
| BCA,1 | FFB,5 | DJG,7 | HIC,8 | EDJ,6 | CHD,9 | AEF,3 | JGI,4 | IAE,2 | GBH,0 |
| IFB,2 | GEF,0 | ECA,6 | DBH,7 | JHD,4 | ADJ,3 | FIC,5 | CAE,9 | BJG,1 | HGI,8 |
| AJG,3 | CCA,9 | FHD,5 | EFB,6 | DGI,7 | BEF,1 | IDJ,2 | HBH,8 | GIC,0 | JAE,4 |
| JIC,4 | DAE,7 | HFB,8 | ACA,3 | IEF,2 | FBH,5 | BHD,1 | GDJ,0 | EGI,6 | CJG,9 |
| GHD,0 | AGI,3 | IIC,2 | JEF,4 | FJG,5 | EAE,6 | CBH,9 | DCA,7 | HDJ,8 | BFB,1 |
| CGI,9 | EIC,6 | GAE,0 | IHD,2 | BBH,1 | HJG,8 | JCA,4 | AFB,3 | DEF,7 | FDJ,5 |

Table 9

Square Number 8

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| CBB,8 | IJF,0 | JAJ,7 | EFG,2 | FHA,1 | AIE,5 | BCC,6 | GEI,4 | HGH,9 | DDD,3 |
| HFG,9 | EBB,2 | FDD,1 | BEI,6 | IGH,0 | GCC,4 | CIE,8 | DJF,3 | AAJ,5 | JHA,7 |
| IDD,0 | GAJ,4 | ABB,5 | HCC,9 | JFG,7 | BGH,6 | DEI,3 | EHA,2 | FIE,1 | CJF,8 |
| DAJ,3 | AEI,5 | ECC,2 | FBB,1 | BIE,6 | CHA,8 | IFG,0 | HDD,9 | JJF,7 | GGH,4 |
| EGH,2 | HIE,9 | CEI,8 | IHA,0 | DBB,3 | FJF,1 | JDD,7 | ACC,5 | GFG,4 | BAJ,6 |
| GIE,4 | BDD,6 | DGH,3 | CAJ,8 | AJF,5 | JBB,7 | HHA,9 | FFG,1 | EEI,2 | ICC,0 |
| JEI,7 | FGH,1 | HJF,9 | DIE,3 | CCC,8 | EDD,2 | GBB,4 | IAJ,0 | BHA,6 | AFG,5 |
| AHA,5 | CFG,8 | IIE,0 | JGH,7 | GDD,4 | HAJ,9 | EJF,2 | BBB,6 | DCC,3 | FEI,1 |
| BJF,6 | JCC,7 | GHA,4 | ADD,5 | HEI,9 | DFG,3 | FAJ,1 | CGH,8 | IBB,0 | EIE,2 |
| FCC,1 | DHA,3 | BFG,6 | GJF,4 | EAJ,2 | IEI,0 | AGH,5 | JIE,7 | CDD,8 | HBB,9 |

Table 10

Square Number 9

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| BFI,9 | JAH,5 | IJC,6 | FGA,4 | DCG,8 | HED,1 | ADF,7 | EBJ,3 | CHE,0 | GIB,2 |
| CGA,0 | FFI,4 | DIB,8 | ABJ,7 | JHE,5 | EDF,3 | BED,9 | GAH,2 | HJC,1 | ICG,6 |
| JIB,5 | EJC,3 | HFI,1 | CDF,0 | IGA,6 | AHE,7 | GBJ,2 | FCG,4 | DED,8 | BAH,9 |
| GJC,2 | HBJ,1 | FDF,4 | DFI,8 | AED,7 | BCG,9 | JGA,5 | CIB,0 | IAH,6 | EHE,3 |
| FHE,4 | CED,0 | BBJ,9 | JCG,5 | GFI,2 | DAH,8 | IIB,6 | HDF,1 | EGA,3 | AJC,7 |
| EED,3 | AIB,7 | GHE,2 | BJC,9 | HAH,1 | IFI,6 | CCG,0 | DGA,8 | FBJ,4 | JDF,5 |
| IBJ,6 | DHE,8 | CAH,0 | GED,2 | BDF,9 | FIB,4 | EFI,3 | JJC,5 | ACG,7 | HGA,1 |
| HCG,1 | BGA,9 | JED,5 | IHE,6 | EIB,3 | CJC,0 | FAH,4 | AFI,7 | GDF,2 | DBJ,8 |
| AAH,7 | IDF,6 | ECG,3 | HIB,1 | CBJ,0 | GGA,2 | DJC,8 | BHE,9 | JFI,5 | FED,4 |
| DDF,8 | GCG,2 | AGA,7 | EAH,3 | FJC,4 | JBJ,5 | HHE,1 | IED,6 | BIB,9 | CFI,0 |

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