SPECIAL DETERMINANTS FOUND WITHIN GENERALIZED PASCAL TRIANGLES

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That Pascal's triangle and two classes of generalized Pascal triangles, the multinomial coefficient arrays and the convolution arrays formed from sequences of sums of rising diagonals within the multinomial arrays, share sequences of $k \times k$ unit determinants was shown in [1]. Here, sequences of $k \times k$ determinants whose values are binomial coefficients in the k^{th} column of Pascal's triangle or numbers raised to a power given by the $(k-1)^{st}$ triangular numbers are explored.

1. INTRODUCTION

First, we imbed Pascal's triangle in rectangular form in the $n \times n$ matrix $P = (p_{ij})$, where

$$p_{ij} = \binom{i + j - 2}{i - 1} ,$$

$$(1.1) \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

Pascal's triangle in left-justified form can be imbedded in the $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \binom{i - 1}{j - 1},$$

$$(1.2) \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

To avoid confusion, note that when Pascal's triangle is imbedded in matrices throughout this paper, we will number the rows and columns in the usual matrix notation, with the leftmost column the first column. If we refer to Pascal's triangle itself, however, then the leftmost column is the zero$^{\text{th}}$ column, and the top row is the zero$^{\text{th}}$ row. While we are dealing with infinite matrices here, the multiplication of $n \times n$ matrices provides an easily understood presentation. As in [1], compositions of generating functions actually lie at the heart of the proofs.

Let us define an arithmetic progression of the r^{th} order, denoted by $(AP)_r$, as a sequence of numbers whose r^{th} row of differences is a row of constants, but whose $(r-1)^{\text{st}}$ row is not. A row of repeated constants is an $(AP)_0$. The constant in the r^{th} row of differences of an $(AP)_r$ will be called the constant of the progression. That the i^{th} row of Pascal's triangle in rectangular form is an $(AP)_i$ with constant 1 was proved in [1]. We will have need of the following theorem from [1], [2].

Theorem 1.1 (Eves' Theorem). Consider a determinant of order n whose i^{th} row $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an $(AP)_{i-1}$ with constant a_i. Then the value of the determinant is the product $a_1 a_2 \cdots a_n$.

2. BINOMIAL COEFFICIENT DETERMINANT VALUES FROM PASCAL'S TRIANGLE

Return again to matrix P of (1.1). Suppose that we remove the top row and left column, and then evaluate the $k \times k$ determinants containing the upper left corner. Then

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4,$$

and the $k \times k$ determinant has value $(k + 1)$.

Proof is by mathematical induction. Assume that the $(k-1) \times (k-1)$ determinant has value k. In the $k \times k$ determinant, subtract the preceding column from each column successively for $j = k, k-1, k-2, \cdots, 2$. Then subtract the preceding row from each row successively for $i = k, k-1, k-2, \cdots, 2$, leaving

$$\begin{vmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & \cdots \\ 0 & 3 & 6 & 10 & \cdots \\ 0 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= 1 + k.$$

Returning to matrix P, take 2×2 determinants along the 2nd and 3rd rows:

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \quad \cdots,$$

giving the values found in the second column of Pascal's left-justified triangle, for

$$\begin{vmatrix} \binom{j}{1} & \binom{j+1}{1} \\ \binom{j+1}{2} & \binom{j+2}{2} \end{vmatrix} = \binom{j+1}{2}$$

by simple algebra. Of course, 1×1 determinants along the second row of P yield the successive values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10\,,$$

the successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the 2nd, 3rd, $\cdots$, and $(k+1)^{st}$ rows yields the successive entries of the k^{th} column of Pascal's triangle.

To formalize our statement,

Theorem 2.1. The determinant of the $k \times k$ matrix $R(k,j)$ taken with its first column the j^{th} column of P, the rectangular form of Pascal's triangle imbedded in a matrix, and its first row the second row of P, is the binomial coefficient

$$\binom{j-1+k}{k}\,.$$

To illustrate,

$$\det R(4,3) = \begin{vmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 6 & 4 & 5 & 6 \\ 10 & 10 & 15 & 21 \\ 15 & 20 & 35 & 56 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 \\ 0 & 6 & 10 & 15 \\ 0 & 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \\ 5 & 15 & 35 & 70 \end{vmatrix}$$

$$= \det R(3,3) + \det R(4,2)$$

$$= \binom{5}{3} + \binom{5}{4} = \binom{6}{4} = 15\,.$$

First, the preceding column was subtracted from each column successively, $j = k, k - 1, \cdots, 2$, and then the preceding row was subtracted from each row successively for $i = k, k - 1, \cdots, 2$. Then the determinant was made the sum of two determinants, one bordering $R(3,3)$ and the other equal to $R(4,2)$ by adding the j^{th} column to the $(j+1)^{st}$, $j = 1, 2, \cdots, k - 1$.

By following the above procedure, we can make

$$\det R(k,j) = \det R(k-1, j) + \det R(k, j-1) .$$

We have already proved that

$$\det R(k,1) = 1 = \binom{k+0}{k}, \qquad \det R(k,2) = k + 1 = \binom{k+1}{k} \quad \text{for all } k,$$

$$\det R(1,j) = j = \binom{j+0}{1}, \qquad \det R(2,j) = \binom{j+1}{2} \quad \text{for all } j .$$

If

$$\det R(k-1,j) = \binom{j+k-2}{k-1} \qquad \text{and} \qquad \det R(k, j-1) = \binom{j+k-2}{k} ,$$

then

$$\det R(k,j) = \binom{j+k-2}{k-1} + \binom{j+k-2}{k} = \binom{j+k-1}{k}$$

for all k and all j by mathematical induction.

Since P is its own transpose, Theorem 2.1 is also true if the words "column" and "row" are everywhere exchanged.

Consider Pascal's triangle in the configuration of A^T, which is just Pascal's rectangular array P with the i^{th} row moved $(i - 1)$ spaces right, $i = 1, 2, 3, \cdots$. Form $k \times k$ matrices $R'(k,j)$ such that the first row of $R'(k,j)$ is the second row of A^T beginning with the j^{th} column of A^T. Then $AR'(k, j-1) = R(k,j)$ as can be shown by considering their column generating functions, and since $\det A = 1$, $\det R'(k, j-1) = \det R(k,j)$, leading us to the following theorems.

Theorem 2.2. Let A^T be the $n \times n$ matrix containing Pascal's triangle on and above its main diagonal so that the rows of Pascal's triangle are placed vertically. Any $k \times k$ submatrix of A^T selected with its first row along the second row of A^T and its first column the j^{th} column of A^T, has determinant value

$$\binom{k+j-2}{k} .$$

Since A is the transpose of A^T, wording Theorem 2.2 in terms of the usual Pascal triangle provides the following.

Theorem 2.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the i^{th} row has determinant value given by the binomial coefficient

$$\binom{k + i - 1}{k} .$$

Returning to the rectangular Pascal matrix P, in Theorem 2.1, the first row of P was omitted to form the $k \times k$ matrix considered. Now we omit any one row.

Theorem 2.4. Let $R_i(k, j)$ be the $k \times k$ matrix formed from the rectangular Pascal matrix P so that its first k rows are the first $(k + 1)$ rows of P with the i^{th} row omitted, and its first column is the j^{th} column of P. Then $\det R_i(k, j)$ is given by the binomial coefficient

$$\binom{j - 1 + k}{k - i + 1} .$$

Proof. Notice that $R_1(k, j) = R(k, j)$ of Theorem 2.1. If the first row is not the row omitted, by successively subtracting the p^{th} column of $R_s(k, j)$ from the $(p + 1)^{st}$ column, $p = k - 1, k - 2, \cdots, 1$, the new array is $R_{s-1}(k - 1, j + 1)$ bordered by a first row with first element one and all others zero, so that

$$\det R_s(k, j) = \det R_{s-1}(k - 1, j + 1) .$$

If the theorem holds when $i = s - 1$, then

$$\det R_{s-1}(k - 1, j + 1) = \binom{(j + 1) - 1 + (k - 1)}{(k - 1) - (s - 1) + 1} = \binom{j - 1 + k}{k - s + 1} = \det R_s(k, j) ,$$

completing a proof by mathematical induction.

3. OTHER DETERMINANTS WITH SPECIAL VALUES

Suppose we form a matrix using the zero^{th}, second, fourth, $\cdots$, $(2r)^{th}$, $\cdots$, rows of Pascal's triangle written in rectangular form. Then the columns contain even subscripted elements only. Since the i^{th} column is still an i^{th} order arithmetic progression, Eves' Theorem should apply. The constant for the j^{th} column will be 2^{j-1}, rather than 1, and the determinant of such a $k \times k$ matrix will be $2^0 \cdot 2^1 \cdot 2^2 \cdots \cdot 2^{k-1}$ or $2^{[k(k-1)/2]}$. To clarify this, we present such a 5×5 matrix below, which has determinant $2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4 = 2^{10}$.

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 5 & 15 & 35 & 70 \\ 1 & 7 & 28 & 84 & 210 \\ 1 & 9 & 45 & 165 & 495 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 2 & 9 & 25 & 55 \\ 0 & 2 & 13 & 49 & 140 \\ 0 & 2 & 17 & 81 & 285 \end{bmatrix}
$$

$$
\approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 16 & 41 \\ 0 & 0 & 4 & 24 & 85 \\ 0 & 0 & 4 & 32 & 145 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 7 & 17 \\ 0 & 0 & 0 & 8 & 44 \\ 0 & 0 & 0 & 8 & 60 \end{bmatrix}
$$

$$
\approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 7 & 17 \\ 0 & 0 & 0 & 8 & 44 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix} .
$$

In this section, we will prove the following:

Theorem 3.1. Form an $n \times n$ matrix in the upper left corner of Pascal's triangle in rectangular form (or in left-justified form) using the rows which are multiples of r so that the $(i+1)^{st}$ row in the matrix is the first n entries of the $(ir)^{th}$ row of the Pascal array, $i = 0, 1, 2, \cdots, n-1$. The determinant of the matrix is $r^{[n(n-1)/2]}$.

To prove this theorem, we require more information about r^{th} order arithmetic progressions.

Lemma 3.1. Let $\{c_j\}$, $j = 0, 1, 2, \cdots$, be a sequence of consecutive elements of an $(AP)_i$ with constants a_i. Then the k^{th} difference sequence has elements given by

$$
\sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{j-p}, \quad \text{and} \quad \sum_{p=0}^{i} (-1)^p \binom{i}{p} c_{j-p} = a_i .
$$

Proof. We list successive differences:

1st: $$c_j - c_{j-1}$$

2nd: $$(c_j - c_{j-1}) - (c_{j-1} - c_{j-2}) = c_j - 2c_{j-1} + c_{j-2}$$

3rd: $$(c_j - 2c_{j-1} + c_{j-2}) - (c_{j-1} - 2c_{j-2} + c_{j-3})$$
$$= c_j = 3c_{j-1} + 3c_{j-2} - c_{j-3} .$$

If the $(k-1)^{st}$ difference has the form of Lemma 3.1, then the k^{th} difference is given by

$$\sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} - \sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p}$$

$$= c_j + \sum_{p=1}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} + \sum_{p=1}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} + (-1)^k c_{j-k}$$

$$= \sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{j-p} \quad ,$$

which establishes the form given in the Lemma. When $k = i$, then the i^{th} difference is the constant of the sequence.

Now form an $n \times n$ matrix P^* with its $(i+1)^{st}$ row the first n entries of the $(ri)^{th}$ row of Pascal's triangle in rectangular form. Then the elements in its $(k+1)^{st}$ column are given by

$$\binom{(r-i)k}{k} .$$

The k^{th} difference sequence for these elements is

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{r-p} = \sum_{p=0}^{k} (-1)^p \binom{k}{p} \binom{(r-p)k}{k} = r^k$$

applying Lemma 3.1 and a formula given by Knuth [3]. The $(k-1)^{st}$ difference is not a constant, however, so that the sequence is an $(AP)_k$. By Eves' Theorem, the determinant, then, will be $r^0 \cdot r^1 \cdot r^2 \cdots \cdot r^{n-1} = r^{n(n-1)/2}$.

If an $n \times n$ matrix A^* is formed using only the $(ri)^{th}$ rows of Pascal's left-justified triangle, and A^T is the transpose of A defined in (1.2), then $A^*A^T = P^*$, since the row generators of A^* are $(1+x)^{r(i-1)}$, and of A^T,

$$\left(\frac{1}{1-x}\right) \cdot \left(\frac{x}{1-x}\right)^{i-1} ,$$

making the row generators of A^*A^T

$$\left(\frac{1}{1-x}\right) \cdot \left(1 + \frac{x}{1-x}\right)^{r(i-1)} = \left(\frac{1}{1-x}\right)^{r(i-1)+1}$$

which we recognize as the row generators of P^*, making $\det A^* = \det P^* = r^{n(n-1)/2}$. (Here, we apply the method of proof using generating functions as in [1].)

For example, when $n = 4$ and $r = 2$, A^*A^T becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 4 & 6 & 4 \\ 1 & 6 & 15 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 5 & 15 & 35 \\ 1 & 7 & 28 & 84 \end{bmatrix}.$$

In fact, we have the same results if the row numbers taken to form P^* or A^* form an arithmetic progression.

Theorem 3.2. Form an $n \times n$ matrix which has its $(i + 1)^{st}$ row the first n entries of the $(ri + s)^{th}$ row of Pascal's triangle in rectangular form, $s \geq 0$, $i = 0, 1, \cdots, n - 1$. The determinant of the matrix is $r^{n(n-1)/2}$.

Proof. Subtract the $(k - 1)^{st}$ column from the k^{th} column for $k = n, n - 1, \cdots, 2$. Repeating this process s times gives the matrix P^*.

Reapplying Eves' Theorem, Theorem 3.1 can be extended to the following.

Theorem 3.3. Form an $n \times n$ matrix such that its $(i + 1)^{st}$ row consists of any n successive entries whose subscripts differ by r from the i^{th} row of Pascal's triangle written in rectangular form, $i = 0, 1, \cdots, n-1$. The determinant of the matrix is $r^{n(n-1)/2}$.

The theorems of this section are special cases of the more general theorem which follows.

Theorem 3.4. Form an $n \times n$ matrix which has its $(i + 1)^{st}$ row the subsequence $\{c_{ir+s}\}$, s arbitrary, of an $(AP)_i$ $\{c_i\}$ with constant a_i, $i = 0, 1, \cdots, n - 1$. Then the determinant is $r^{n(n-1)/2} a_0 a_1 a_2 \cdots a_{n-1}$.

The proof, which is omitted, hinges upon showing that $\{c_{ir+s}\}$ is an $(AP)_i$ with constant $r^i a_i$ and applying Eves' Theorem.

The theorems of this section can also be stated for columns. Next, the results can be extended to convolution arrays and to multinomial coefficient arrays by considering certain matrix products.

4. MULTINOMIAL COEFFICIENT ARRAYS

Let the array of multinomial coefficients arising from expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, be called the m-multinomial coefficient array. Let the left-justified m-multinomial coefficient array be imbedded in an $n \times n$ matrix A_m. Let the $n \times n$ matrix F_m contain the rows of A_m as the columns of F_m written on and below the main diagonal. Let A^T be the transpose of the $n \times n$ matrix A of (1.2). Then the matrix equation

$$F_{m-1}A^T = A_m^T, \qquad m = 2, 3, \cdots$$

was proved in [1]. Since any $k \times k$ submatrix of A_m^T having its first row along the second row of A_m^T and its first column the j^{th} column of A_m^T is the product of a submatrix of

F_{m-1} with a unit determinant and a $k \times k$ submatrix of A^T satisfying Theorem 2.2, its determinant will be given by

$$\binom{k + j - 2}{k}.$$

Since the transpose of A_m^T is A_m, we restate these results in terms of the m-multinomial coefficient array.

Theorem 4.1. The determinant of the $k \times k$ matrix formed with its first column the first column of any multinomial coefficient array in left-justified form (the column of successive whole numbers) and its first row the i^{th} row of that multinomial coefficient array, has value given by the binomial coefficient

$$\binom{k + i - 1}{k}.$$

Now let $(A^*)^T$ be the transpose of the $n \times n$ matrix A^* formed with its $(i + 1)^{st}$ row the first n entries of the $(ri)^{th}$ row of Pascal's left-justified triangle, $i = 0, 1, \cdots, n - 1$. Then the matrix product $F_{m-1}(A^*)^T = (A_m^*)^T$, where A_m^* is the $n \times n$ matrix formed using only the $(ri)^{th}$ rows of the m-multinomial coefficient array, $i = 0, 1, \cdots, n - 1$, as can be proved by examining the column generating functions. For, the column generators of F_{m-1} are $G_j(x) = [x(1 + x + \cdots + x^{m-1})]^{j-1}$ and of $(A^*)^T$, $H_j(x) = (1 + x)^{r(j-1)}$, $j = 1, 2, \cdots, n$, making the column generators of $F_{m-1}(A^*)^T$ be $H_j(G_j(x)) = (1 + x + x^2 + \cdots + x^m)^{r(j-1)}$, which we recognize as the column generators for the matrix $(A_m^*)^T$ claimed above. Again, considering the very special products of submatrices involved, we are led to the following result.

Theorem 4.2. If any multinomial coefficient array is written in left-justified form, the determinant of the $k \times k$ matrix formed with its $(i + 1)^{st}$ row the first n entries of the $(ri)^{th}$ row of the multinomial array, $i = 0, 1, 2, \cdots, k - 1$, is given by $r^{k(k-1)/2}$.

As an example, for $n = 5$ and $r = 2$, $F_1(A^*)^T = (A_2^*)^T$ becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 1 & 6 & 15 & 28 \\ 0 & 0 & 4 & 20 & 56 \\ 0 & 0 & 1 & 15 & 70 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 3 & 10 & 21 & 36 \\ 0 & 2 & 16 & 50 & 112 \\ 0 & 1 & 19 & 90 & 266 \end{bmatrix}$$

where $(A_2^*)^T$ has alternate rows of the trinomial coefficient array appearing as its columns, and the determinant equals 2^{10}.

Further examination of a matrix product, $F_{m-1}(A_m^{**})^T$, where the $n \times n$ matrix A_m^{**} is formed with its $(i+1)^{st}$ row the first n entries of the $(ri + s)^{th}$ row of the m-multinomial coefficient array, $s \geq 0$, $i = 0, 1, 2, \cdots, n - 1$, shows that Theorem 3.2 can be extended to the multinomial coefficients.

Theorem 4.3. Consider any left-justified multinomial coefficient array. Form a $k \times k$ matrix with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of the multinomial array, $i = 0, 1, \cdots, k-1$, $s \geq 0$. The determinant of that matrix is given by $r^{k(k-1)/2}$.

5. THE FIBONACCI CONVOLUTION ARRAY AND RELATED ARRAYS

The Fibonacci sequence, when convolved with itself $j-1$ times, forms the sequence in the j^{th} column of the matrix C below (see [1] and [4])

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 3 & 10 & 22 & 40 & 65 & 98 & \cdots \\ 5 & 20 & 51 & 105 & 190 & 315 & \cdots \\ 8 & 38 & 111 & 256 & 511 & 924 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the original sequence is in the leftmost column and the column generators are given by $[1/(1-x-x^2)]^j$, $j = 1, 2, \cdots, n$. If F_1 is the $n \times n$ matrix formed as in Section 3 with the rows of Pascal's triangle in vertical position on and below the main diagonal, and P is Pascal's rectangular array (1.1), then $F_1P = C$ as proved in [1]. Now, since here submatrices of C taken along the second row of C are the product of submatrices of F_1 with unit determinants and similarly placed submatrices of P whose determinants are given in Theorem 2.1, these submatrices of C have determinant values given by the same binomial coefficients found for P.

The generalization to convolution triangles for sequences which are found as sums of rising diagonals of m-multinomial coefficient arrays written in left-justified form is not difficult, since the matrix product F_mP yields just those arrays as shown in [1]. We thus write the following theorem.

Theorem 5.1. Let the convolution triangle for the sequences of sums found along the rising diagonals of the left-justified m-multinomial coefficient array be written in rectangular form and imbedded in an $n \times n$ matrix C^*. Then the determinant of any $k \times k$ submatrix of C^* selected with its first row along the second row of C^* and its first column the j^{th} column of C^* has determinant value given by the binomial coefficient

$$\binom{k+j-1}{k} .$$

Now, let P^* be the $n \times n$ matrix with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of Pascal's rectangular array P, $i = 0, 1, \cdots, n-1$, $s \geq 0$. Paralleling the development given for Theorem 5.1 but considering the matrix product F_mP^* which is the $n \times n$ matrix containing the $(ri + s)^{th}$ rows of the convolution array for the rising diagonals of the given m-multinomial coefficient array, we find that Theorem 3.2 extends to the following.

Theorem 5.2. If a $k \times k$ matrix is formed with its $(i + 1)^{st}$ row the first n entries of the $(ri + s)^{th}$ row of the rectangular convolution array for the rising diagonals of any left-justified multinomial coefficient array, $i = 0, 1, \cdots, k - 1$, $s \geq 0$, then its determinant has value $r^{k(k-1)/2}$.

Lastly, consider the sequence of sums of elements $u_m(n; p, 1)$ found on the rising diagonals formed by beginning at the leftmost column of a left-justified m-multinomial coefficient array and going up p and to the right one throughout the array. (For the Pascal triangle, these numbers are the generalized Fibonacci numbers $u(n; p, 1)$ of Harris and Styles [5].) Form the matrix $D_m(p,1)$ so that the sequence of elements having $u_m(n; p, 1)$ as its sum lies (in reverse order) on its rows. $D_m(p,1)$ will have a one for each element on its main diagonal and each column will contain the corresponding row of the m-multinomial array but with $(p - 1)$ zeros between entries, so that the generating functions for its columns are $[x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^j$, $j = 1, 2, \cdots, n$. It was shown in [1] that $D_m(p,1)P$ gives the convolution triangle in rectangular form for the sequence $u_m(n; p, 1)$. By examining the column generators, we also have that $D_m(p,1)P^*$ gives the array containing the $(ir + s)^{th}$ rows of the convolution triangle for the sequence $u_m(n, p, 1)$. Putting all of this together, we write the following theorem.

Theorem 5.3. Write the convolution triangle in rectangular form imbedded in an $n \times n$ matrix C_m^* for the sequence of sums found on the rising diagonals formed by beginning at the leftmost column and moving up p and right one throughout any left-justified multinomial coefficient array. The $k \times k$ submatrix formed with its first row the second row of C_m^* and its first column the j^{th} column of C_m^* has determinant given by the binomial coefficient

$$\binom{k + j - 1}{k} .$$

The $n \times n$ matrix formed with its $(i + 1)^{st}$ row the first n entries of the $(ri + s)^{th}$ row of the convolution triangle, $i = 0, 1, \cdots, n - 1$, $s \geq 0$, has determinant equal to $r^{n(n-1)/2}$.

REFERENCES

1. Marjorie Bicknell and V. E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 11, No. 2, pp. 131-144.
2. Howard Eves in a letter to V. E. Hoggatt, Jr., January, 1972.
3. Donald E. Knuth, Fundamental Algorithms, Vol. 1, Addison-Wesley, Menlo Park, Calif., 1969, p. 64, Eq. 36, Sec. (1.2.6).
4. Verner E. Hoggatt, Jr., and Marjorie Bicknell, "Convolution Triangles," Fibonacci Quarterly, Vol. 10, No. 6, pp. 599-608.
5. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 4, pp. 277-289.