GENERALIZED FIBONACCI POLYNOMIALS

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The Fibonacci polynomials and their relationship to diagonals of Pascal's triangle are generalized in this paper. The generalized Q-matrix investigated by Ivie [1] occurs as a special case.

1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials, defined by

$$(1.1)\qquad F_0(x) = 0, \quad F_1(x) = 1, \quad F_2(x) = x, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

are well known to readers of this journal. That the Fibonacci polynomials are generated by a matrix Q_2,

$$(1.2)\qquad Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix},$$

can be verified quite easily by mathematical induction. Also, it is apparent that, when $x = 1$, $F_n(1) = F_n$, the n^{th} Fibonacci number, and when $x = 2$, $F_n(2) = P_n$, the n^{th} Pell number.

Further, when Pascal's triangle is written in left-justified form, the sums of the elements along the rising diagonals give rise to the Fibonacci numbers, and, in fact, those elements are the coefficients of the Fibonacci polynomials. That is,

$$(1.3)\qquad F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1},$$

where $[x]$ is the greatest integer contained in x, and

is a binomial coefficient $\binom{n}{j}$

The first few Fibonacci polynomials are displayed below as well as the array of their coefficients.

Fibonacci Polynomials	Coefficient Array				
$F_1(x) = 1$	1				
$F_2(x) = x$	1				
$F_3(x) = x^2 + 1$	1	1			
$F_4(x) = x^3 + 2x$	1	2			
$F_5(x) = x^4 + 3x^2 + 1$	1	3	1		
$F_6(x) = x^5 + 4x^3 + 3x$	1	4	3		
$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$	1	5	6	1	
$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$	1	6	10	4	
$\cdots \quad \cdots$	...	...	...	...	...

If one observes that, by rule of formation of the Fibonacci polynomials, if one writes the polynomials in descending order, to form the coefficient of the k^{th} term of $F_n(x)$, one adds the coefficients of the k^{th} term of $F_{n-1}(x)$ and the $(k-1)^{st}$ term of $F_{n-2}(x)$, the array of coefficients formed has the same rule of formation as Pascal's triangle when it is written in left-justified form, except that each column is moved one line lower, so that the coefficients formed are those elements that appear along the diagonals formed by beginning in the left-most column and preceding up one and right one throughout the left-justified Pascal triangle. Throughout this paper, this diagonal will be called the rising diagonal of such an array.

2. THE TRIBONACCI POLYNOMIALS

Define the Tribonacci polynomials by

$$T_{-1}(x) = T_0(x) = 0, \qquad T_1(x) = 1, \qquad T_2(x) = x^2,$$

$$T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x). \tag{2.1}$$

When $x = 1$, $T_n(1) = T_n$, the n^{th} Tribonacci number $1, 1, 2, 4, 7, 13, 24, 44, 81, \cdots$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. The first few Tribonacci polynomials follow.

Tribonacci Polynomials

$$T_1(x) = 1$$

$$T_2(x) = x^2$$

$$T_3(x) = x^4 + x$$

$$T_4(x) = x^6 + 2x^3 + 1$$

$$T_5(x) = x^8 + 3x^5 + 3x^2$$

$$T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$$

$$T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1$$

$$T_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2$$

Tribonacci Coefficient Array

1				
1				
1	1			
1	2	1		
1	3	3		
1	4	6	2	
1	5	10	7	1
1	6	15	16	6
...	...	...	...	...

Left-Justified Trinomial Coefficient Array

$(1 + x + x^2)^n, \qquad n = 0, 1, 2, \cdots$

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
...	...	...	...	...	...	...	...	...	...	...	...	...

The Tribonacci coefficient array has the same rule of formation as the trinomial coefficient array, except that each column is placed one line lower. Thus, the sums of the rows are the same as the sums of the rising diagonals of the trinomial coefficient array, both sums yielding the Tribonacci numbers 1, 1, 2, 4, 7, 13, 24, $\cdots$, and the coefficients of the Tribonacci polynomials are the trinomial coefficients found on those same rising diagonals. That is,

$$T_n(x) = \sum_{j=0} \binom{n-j-1}{j}_3 x^{2n-3j-2}, \tag{2.2}$$

where

$$\binom{n}{j}_3$$

is the trinomial coefficient in the n^{th} row and j^{th} column where, as is usual, the left-most column is the zero$^{\text{th}}$ column and the top row the zero$^{\text{th}}$ row, and

$$\binom{n}{j}_3 = 0 \quad \text{if} \quad j > n.$$

The Tribonacci polynomials are generated by the matrix Q_3,

$$Q_3 = \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

so that

$$(2.3) \qquad Q_3^n = \begin{pmatrix} T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ xT_n(x) + T_{n-1}(x) & xT_{n-1}(x) + T_{n-2}(x) & xT_{n-2}(x) + T_{n-3}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{pmatrix} .$$

A proof could be made by mathematical induction. That Q_3^n has the given form for $n = 1$ is apparent by inspection of Q_3, element-by-element. Expansion of the matrix product $Q_3^{n+1} = Q_3 Q_3^n$ gives the elements of Q_3^{n+1} in the required form, making use of the recursion (2.1).

Notice that $\det Q_3^n = 1^n = 1$, analogous to the Fibonacci case. In fact, we can write an interesting determinant identity. Again using (2.1), we multiply row one of Q_3^n by x^2 and add to row 2. Then we exchange rows 1 and 2 to write

$$(2.4) \qquad (-1) = \begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{vmatrix} ,$$

which becomes an identity for Tribonacci numbers when $x = 1$.

3. THE QUADRANACCI POLYNOMIALS

The Quadranacci polynomials are defined by $T^*_{-2}(x) = T^*_{-1}(x) = T^*_0(x) = 0$, $T^*_1(x) = 1$,

$$(3.1) \qquad T^*_{n+4}(x) = x^3 T^*_{n+3}(x) + x^2 T^*_{n+2}(x) + xT^*_{n+1}(x) + T^*_n(x) .$$

The first few values are

$$\begin{aligned}
T_1(x) &= 1 \\
T_2(x) &= x^3 \\
T_3(x) &= x^6 + x^2 \\
T_4(x) &= x^9 + 2x^5 + x \\
T_5(x) &= x^{12} + 3x^8 + 3x^4 + 1 \\
T_6(x) &= x^{15} + 4x^{11} + 6x^7 + 4x^3 \\
T_7(x) &= x^{18} + 5x^{14} + 10x^{10} + 10x^6 + 3x^2 \\
\cdots &\quad \cdots
\end{aligned}$$

Notice that the coefficient of the j^{th} term of $T_n^*(x)$ is the sum of the coefficients of the j^{th} term of $T_{n-1}^*(x)$, $(j-1)^{st}$ term of $T_{n-2}^*(x)$, $(j-2)^{nd}$ term of $T_{n-3}^*(x)$, and $(j-3)^{rd}$ term of $T_{n-4}^*(x)$ when the polynomials are arranged in descending order. Then, the array of coefficients, if each row were moved up one line, would have the same rule of formation as the left-justified array of quadranomial coefficients, arising from expansions of $(1+x+x^2+x^3)^n$, $n = 0, 1, 2, \cdots$. Thus, the coefficients of $T_n^*(x)$ are those found on the n^{th} rising diagonal of the quadranomial triangle. Also, $T_n^*(1) = T_n^*$, the n^{th} Quadranacci number 1, 1, 2, 4, 8, 15, 29, 56, 108, $\cdots$, $T_{n+4}^* = T_{n+3}^* + T_{n+2}^* + T_{n+1}^* + T_n^*$.

The Quadranacci polynomials are generated by the matrix Q_4,

$$Q_4 = \begin{pmatrix} x^3 & 1 & 0 & 0 \\ x^2 & 0 & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,$$

so that $Q_4^n = (a_{ij})$ has its j^{th} column given by $a_{1j} = T_{n+2-j}^*(x)$, $a_{2j} = x^2T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-1-j}^*(x)$, $a_{3j} = xT_{n+1-j}^*(x) + T_{n-j}^*(x)$, and $a_{4j} = T_{n+1-j}^*(x)$, $j = 1, 2, 3, 4$. That Q_4^n has the form claimed above can be established by mathematical induction. That Q_4 has the stated form follows by inspection. Let $Q_4^{n+1} = (b_{ij})$ and $Q_4 = (q_{ij})$. Then we expand $Q_4^{n+1} = Q_4 Q_4^n$. The first row of Q_4^{n+1} has the required form, for

$$\begin{aligned} b_{1j} &= q_{11}a_{1j} + q_{12}a_{2j} + q_{13}a_{3j} + q_{14}a_{4j} \\ &= [x^3 T_{n+2-j}^*(x)] + [x^2 T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-1-j}^*(x)] + 0 + 0 \\ &= T_{(n+1)+2-j}^* \quad , \end{aligned}$$

where we make use of (3.1). Computation of b_{2j}, b_{3j}, and b_{4j} is similar, and shows that Q_4^{n+1} has the required form, which would complete the proof.

We derive a determinant identity for Quadranacci polynomials from Q_4^n by forming the matrix Q_4^{*n} as follows. Add x^3 times row 1 to row 2, making $a_{2j}' = T_{n+3-j}^*(x)$. Add x^2 times row 1 and x^3 times row 2 to row 3, producing $a_{3j}' = T_{n+4-j}^*(x)$. Exchange rows 1 and 3. Then matrix Q_4^{*n} has

$$a_{1j}^* = T_{n+4-j}^*(x), \quad a_{2j}^* = T_{n+3-j}^*(x), \quad a_{3j}^* = T_{n+2-j}^*(x) ,$$

and

$$a_{4j}^* = T_{n+1-j}^*(x), \quad j = 1, 2, 3, 4, \quad \text{and} \quad \det Q_4^{*n} = (-1)^{n+1}$$

because there was one row exchange. That is, for example, when $x = 1$,

$$(3.2) \qquad (-1)^{n+1} = \begin{vmatrix} T^*_{n+3} & T^*_{n+2} & T^*_{n+1} & T^*_{n} \\ T^*_{n+2} & T^*_{n+1} & T^*_{n} & T^*_{n-1} \\ T^*_{n+1} & T^*_{n} & T^*_{n-1} & T^*_{n-2} \\ T^*_{n} & T^*_{n-1} & T^*_{n-2} & T^*_{n-3} \end{vmatrix},$$

where T^*_n is the n^{th} Quadranacci number.

4. THE R-BONACCI POLYNOMIALS

Define the r-bonacci polynomials by

$$R_{-(r-2)}(x) = R_{-(r-1)}(x) = \cdots = R_{-1}(x) = R_0(x) = 0, \quad R_1(x) = 1, \quad R_2(x) = x^{r-1},$$

$$(4.1) \qquad R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \cdots + R_n(x) .$$

The r-bonacci polynomials, by their recursive definition will have the coefficients of $R_n(x)$, written in descending order, given by the coefficients on the n^{th} rising diagonal of the left-justified r-nomial coefficient array, the coefficients arising from expansions of

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, \cdots .$$

That is,

$$(4.2) \qquad R_n(x) = \sum_{j=0} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj} ,$$

where

$$\binom{n}{j}_r$$

is the element in the n^{th} row and j^{th} column of the left-justified r-nomial triangle, and

$$\binom{n}{j}_r = 0 \quad \text{when} \quad j > n .$$

The r-bonacci polynomials are generated by the $r \times r$ matrix Q_r,

$$Q_r = \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \cdots & 0 \\ x^{r-2} & 0 & 1 & 0 & \cdots & 0 \\ x^{r-3} & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is an identity matrix of order $(r - 1)$ bordered on the left by a column of descending powers of x and followed by a bottom row of zeros. The matrix Q_r^n has $R_{n+1}(x)$ as the element in the upper left and $R_n(x)$ in the lower left, with general element a_{ij} given by

$$a_{ij} = \sum_{k=1}^{r-1+1} x^{r+1-k-i} R_{n+1-j-k}(x) \quad . \tag{4.3}$$

Proof of (4.3) is by mathematical induction. Let $Q_r = (b_{ij})$. Then $b_{i1} = x^{r-i}$, $i = 1, 2, \cdots, r$; $b_{ij} = 1$, $j = i + 1$, $i = 1, 2, \cdots, r$; and $b_{ij} = 0$ whenever $j \neq 1$ and $j \neq i + 1$. Let

$$Q_r^{n+1} = Q_r Q_r^n = (c_{ij}) \quad .$$

Then

$$\begin{aligned} c_{ij} = \sum_{k=1}^{r} b_{ik} a_{kj} &= b_{i1} a_{1k} + \sum_{k=2}^{r} b_{ik} a_{kj} \\ &= x^{r-i} R_{n+1-j}(x) + a_{i+1,j} + 0 \\ &= x^{r-i} R_{n+1-j}(x) + \sum_{k=1}^{r-i} x^{r-k-i} R_{n+1-j-k}(x) \\ &= \sum_{k=1}^{r-i+1} x^{r+1-k-i} R_{(n+1)+1-j-k}(x) \quad , \end{aligned}$$

which is the required form for the general element of Q_r^{n+1}, completing the proof.

If we operate upon Q_r^n as before, we can again make a determinant identity. Repeatedly add x^{r-1} times the $(i - 1)^{st}$ row, x^{r-2} times the $(i - 2)^{nd}$ row, $\cdots$, x^{r+1-i} times the first row to the i^{th} row, to produce a new i^{th} row with R_{n+i-1} in its first column, for $i = 2, 3, \cdots, r - 1$. Then make $(r - 1)/2$ row exchanges to put the elements in the columns in descending order. The matrix R formed has its general element given by

$$r_{ij} = R_{n+r+1-i-j}(x) \ ,$$

and its determinant has value $(-1)^{(r-1)n+[(r-1)/2]}$. That is, when $x = 1$, the r-bonacci numbers

$$\cdots, 0, 1, 1, 2, \cdots, R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n \ ,$$

have the determinant identity

$$\det R = \begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & \cdots & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & \cdots & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)(2n+1)/2}$$

Notice that Eq. (1.2) gives

$$\det Q_2^n = F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n = \det R$$

for r = 2. Since we recognize

$$\left| F_{n+1}(x)F_{n-1}(x) - F_n^2(x) \right| = 1$$

as the characteristic value [2], [3], [4] of sequences arising from the Fibonacci polynomials, we define $|\det R| = 1$ as the characteristic value of the sequences arising from the generalized Fibonacci polynomials, $r \geq 2$. Then, for example, (2.4) gives the characteristic value $|(-1)| = 1$ for the sequences arising from the Tribonacci polynomials, while (3.2) is the array giving the characteristic value $|(-1)^{n+1}| = 1$ for the Quadranacci numbers.

The matrix R just defined has the interesting property that multiplication by Q_r^n produces a matrix of the same form. To clarify, let $R = R_{r,n} = (r_{ij})$ be the $r \times r$ matrix with $R_n(x)$ appearing in the lower left corner, $r_{ij} = R_{n+r+1-i-j}(x)$. Then

$$R_{r,0}Q_r^n = R_{r,n} \tag{4.4}$$

which is proved by mathematical induction as follows. Consider the matrix product $R_{r,n}Q_r = (p_{ij})$ for any n, where we observe that the first column of Q_r contains the multipliers for the recursion relation for the polynomials $R_n(x)$. The i^{th} row of $R_{r,n}$ multiplied by the first column of Q_r produces

$$p_{i1} = \sum_{k=1}^{r} R_{n+r+1-i-k}(x)\, x^{r+1-k} = R_{n+r+1-i}(x) = R_{(n+1)+r+1-i-1}(x) ,$$

while, when $j \neq 1$, since the only non-zero elements of Q_r occur when $i = j - 1$, the i^{th} row of $R_{r,n}$ times the j^{th} column of Q_r produces

$$p_{ij} = R_{n+r+1-i-(j-1)} = R_{(n+1)+r+1-i-j}, \qquad j = 2, 3, \cdots, r ,$$

so that $R_{r,n}Q_r = R_{r,n+1}$, for any n. Then, we must have that $R_{n,0}Q_r = R_{r,1}$, and, if

$$R_{r,0}Q_r^{k-1} = R_{r,k-1} ,$$

then

$$R_{r,0}Q_r^k = (R_{r,0}Q_r^{k-1})Q_r = R_{r,k-1}Q_r = R_{r,k} ,$$

which completes a proof of (4.4) by mathematical induction.

If we equate the elements in the upper left corner of $R_{r,n}$ and $R_{r,0}Q_r^n$ we obtain the identity

$$
\begin{aligned}
R_{n+r-1}(x) = R_{r-1}(x)R_{n+1}(x) &+ R_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+2}(x)] \\
&+ R_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+3}(x)] \\
&+ \cdots \\
&+ R_1(x)[xR_n(x) + R_{n-1}(x)] + R_0(x)R_n(x) \ .
\end{aligned}
\tag{4.5}
$$

Notice that the matrix Q_r provides the multipliers for the recursion relation for the polynomials $R_n(x)$ but does not depend upon the original values of the polynomials in the proof of (4.4). Let $H_n(x)$ be any sequence of polynomials with r arbitrary starting values $H_0(x), H_1(x), \cdots, H_{r-1}(x)$, and with the same recursion relation as the polynomials $R_n(x)$. Form the matrix $R^*_{r,n} = (r^*_{ij})$, $r^*_{ij} = H_{n+r+1-i-j}(x)$. By the arguments used earlier, we can derive $R^*_{r,0}Q_r^n = R^*_{r,n}$ and thus obtain

$$
\begin{aligned}
H_{n+r-1} = H_{r-1}(x)R_{n+1}(x) &+ H_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+2}(x)] \\
&+ H_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+3}(x)] \\
&+ \cdots \\
&+ H_1(x)[xR_n(x) + R_{n-1}(x)] + H_0(x)R_n(x) \ .
\end{aligned}
\tag{4.6}
$$

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