FIBONACCI AND APOLLONIUS

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Apollonius proposed the problem: Given three fixed circles, to find a circle which touches all of them. In general, there are eight solutions. Obviously, if the given circles are mutually tangent, the number of solutions is reduced to two. This case is a favorite with problemists for creating puzzlers and formulas have been found for their solution. In this note, we shall consider only the case where the given circles are mutually tangent and have their centers on the vertices of a Pythagorean triangle. The purpose of this note is to point out a relation between these five circles and any four consecutive Fibonacci numbers. Let r_1, r_2, r_3 denote the given radii; R and r denote the required radii; and F_n, F_{n+1}, F_{n+2}, F_{n+3} any four consecutive Fibonacci numbers. Assume $r_1 < r_2 < r_3$ and $R > r$.

For convenience in computation, we shall denote our Fibonacci numbers by a, b, c, d. Then using b, c as generators, we get the Pythagorean triplets:

$$c^2 - b^2; \quad 2bc; \quad c^2 + b^2 .$$

Then by the condition of our problem, we get

$$\begin{aligned} r_1 + r_2 &= c^2 - b^2 \\ r_1 + r_3 &= 2bc \\ r_2 + r_3 &= c^2 + b^2 . \end{aligned}$$

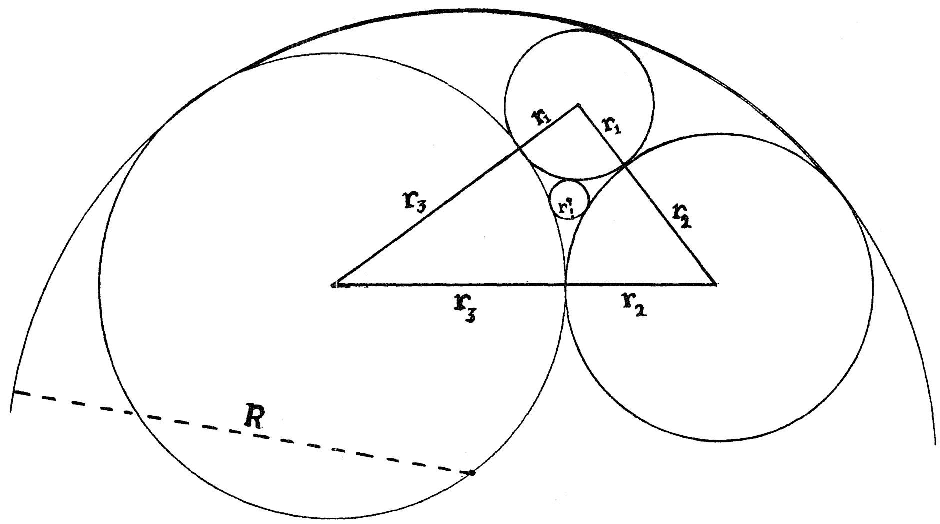
Solving we get

$$\begin{aligned} r_1 &= b(c - b) = ab \\ r_2 &= c(c - b) = ac \\ r_3 &= b(c + b) = bd \\ r_1 + r_2 + r_3 &= c(b + c) = cd . \end{aligned}$$

Then

$$\begin{aligned} r_1 r_2 r_3 &= a^2 b^2 c d \\ r_1 r_2 &= a^2 b c \\ r_1 r_3 &= a b^2 d \\ r_2 r_3 &= a b c d . \end{aligned}$$

The formula below is due to Col. Beard and applies to all cases where the given circles are mutually tangent.



$$R \text{ or } r = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 + 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} .$$

The negative sign gives R (absolute value) and positive sign gives r.

Substituting the values already found for r_1, r_2, r_3 we get

$$R \text{ or } r = \frac{a^2b^2cd}{a^2bc + ab^2d + abcd \mp 2\sqrt{a^2b^2cd \cdot cd}}$$

$$R = \frac{a\,b\,c\,d}{ac + bd - cd} = -c\,d$$

$$r = \frac{a\,b\,c\,d}{4\,cd - ab} .$$

Hence in Fibonacci numbers we have

$$r_1 = F_n F_{n+1}$$

$$r_2 = F_n F_{n+2}$$

$$r_3 = F_{n+1} F_{n+3}$$

$$R = F_{n+2} F_{n+3}$$

$$r = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{4F_{n+2} F_{n+3} - F_n F_{n+1}} .$$

All this holds for Lucas numbers, also.

REFERENCES

1. Kasner and Newman, Mathematics and the Imagination, pp. 13 and 14.
2. Col. R. S. Beard, "A Variation of the Apollonius Problem," Scripta Mathematica, Vol. 21 (1955), pp. 46-47.