NOTE ON A COMBINATORIAL ALGEBRAIC IDENTITY AND ITS APPLICATION

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The identity

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{sr+t}{m} = \begin{cases} 0 & (m < n) \\ (-s)^n & (m = n)\end{cases}, \tag{1}$$

is well-known (cf. Schwatt [4, p. 104] and Gould [3, Formula (3.150)]) and has been utilized by Gould [1], [2] in proving some elegant combinatorial identities, e.g.,

$$\sum_{k=0}^{n}\binom{a+b(n-k)}{n-k}\binom{bk+c}{k}\frac{c}{bk+c} = \binom{a+c+bn}{n}, \tag{2}$$

and

$$\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{k+j}\binom{n}{k}\binom{n}{j}\binom{kj+t}{n} = n!\ . \tag{3}$$

In what follows, we shall establish a combinatorial algebraic identity which involves a wider generalization of (1). We offer the following

Theorem. Let $F(X)$ be a polynomial of degree $m \le n$ in X having the leading term P_0X^m. Then for arbitrary quantities $P_1, \cdots, P_n$ and Q we have

$$F(Q) + \sum_{r=1}^{n}(-1)^r \sum_{1\le k_1<\cdots<k_r\le n} F(P_{k_1}+\cdots+P_{k_r}+Q) = \binom{m}{n}(-1)^n n!\, P_0P_1\cdots P_n, \tag{4}$$

where the inner summation extends over all the r-combinations $(k_1, \cdots, k_r)$ of the integers $1, 2, \cdots, n$, and $\binom{m}{n}$ is 0 or 1 according as $m < n$ or $m = n$.

As a consequence of (4) we have a pair of generalized Euler identities (with $m \le n$) as follows:

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$$\binom{Q}{m} + \sum_{r=1}^{n} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le n} \binom{P_{k_1} + \cdots + P_{k_r} + Q}{m} = \binom{m}{n} (-1)^n P_1 P_2 \cdots P_n , \tag{5}$$

and

$$Q^m + \sum_{r=1}^{n} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le n} (P_{k_1} + \cdots + P_{k_r} + Q)^m = \binom{m}{n} (-1)^n n!\, P_1 P_2 \cdots P_n . \tag{6}$$

Clearly (1) is a special case of (5) with $P_1 = \cdots = P_n = s$. For $P_1 = \cdots = P_n = 1$, $Q = 0$, and $m = n$ we find that (6) implies the familiar Euler theorem about the n^{th} difference of x^n at $x = 0$, viz.

$$\Delta^n 0^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^n = n! .$$

Gould [3, Formula (Z.8)] has remarked about the use of this to determine certain combinatorial identities easily.

With other choices of the P_k's and Q these identities (5) and (6) may give somewhat "strange" but elementary identities such as

$$\sum_{r=1}^{n} (-1)^{n-r} \sum_{1 \le k_1 < \cdots < k_r \le n} \binom{k_1^2 + \cdots + k_r^2}{n} = (n!)^2 \tag{7}$$

and

$$\sum_{r=1}^{n} (-1)^{n-r} \sum_{1 \le k_1 < \cdots < k_r \le n} (k_1^m + \cdots + k_r^m)^n = (n!)^{m+1} . \tag{8}$$

Since every polynomial $F(x)$ of degree m can be expressed as a linear combination of

$$\binom{x}{0}, \binom{x}{1}, \cdots, \binom{x}{m} ,$$

it is easily observed that (4), (5) and (6) are implied by each other. In other words, (4), (5), and (6) are logically equivalent.

For the proof of (4) it suffices to verify (6). Actually (6) can be verified by means of the principle of inclusion and exclusion in combinatorial analysis. Let us expand

$$(P_{i_1} + \cdots + P_{i_r} + Q)^m$$

in accordance with the multinomial theorem and consider a typical term of the form with exponents $a_1 \geq 1, \cdots, a_r \geq 1, \; b \geq 0$:

$$C P_{i_1}^{a_1} \cdots P_{i_r}^{a_r} Q^b, \qquad \left(C = \frac{m!}{a_1! \cdots a_r! \, b!}, \quad a_1 + \cdots + a_r + b = m \right),$$

where $(i_1, \cdots, i_r)$ is an r-subset of $(1, 2, \cdots, n)$. First consider the case $r < n$. In this case the difference set $(j_1, \cdots, j_{n-r}) = (1, 2, \cdots, n) - (i_1, \cdots, i_r)$ is non-empty, so that the typical term occurs in the inner sum

$$(-1)^r \sum (P_{k_1} + \cdots + P_{k_r} + Q)^m$$

and also in all those inner sums of (6) following this one. Consequently, the total number of occurrences of the term is given by

$$(-1)^r \left\{ \binom{n-r}{0} - \binom{n-r}{1} + \binom{n-r}{2} - \cdots + (-1)^{n-r} \binom{n-r}{n-r} \right\} = 0 .$$

This means that every term with $r < n$ vanishes always by cancellation, and this is generally true for $m < n$. For the case $m = n$, the only exceptional term is

$$(-1)^n n! \, P_1 P_2 \cdots P_n Q^0$$

which cannot be cancelled out anyway. Finally, the number of occurrences of the particular term Q^m is seen to be

$$1 - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0 .$$

Thus (6) is completely verified.

Similarly, a direct verification of (5) can be accomplished by using Vandermonde's multiple convolution formula (instead of the multinomial theorem) for expansion of the summands.

APPLICATION

For $m = n$ and $Q = 0$ the identities (5) and (6) imply that every integer $N = P_1 P_2 \cdots P_n$ with n relatively prime factors $P_1, P_2, \cdots, P_n$ can always be represented as an algebraic sum of

$$\binom{\sum P}{n} \text{'s} \quad ,$$

and that $N = n! \, P_1 P_2 \cdots P_n$ as an algebraic sum of the n^{th} powers.

It is known that there are infinitely many solutions of the equation $A^3 + B^3 + C^3 = D^3$ in positive integers (see Shanks [5, p. 157]). Here as a simple application of (6) we shall construct certain sets of non-trivial positive integral solutions of the 2-sided 3-cube equation

$$X_1^3 + X_2^3 + X_3^3 = Y_1^3 + Y_2^3 + Y_3^3 \ . \tag{9}$$

Making use of (6) with $m = n = 3$ and $Q = 0$ we have

$$\begin{aligned} P_1^3 + P_2^3 + P_3^3 + (P_1 + P_2 + P_3)^3 &= (P_1 + P_2)^3 + (P_2 + P_3)^3 \\ &\quad + (P_3 + P_1)^3 + 6P_1P_2P_3 \ . \end{aligned} \tag{10}$$

Let $P_1^3 = 6P_1P_2P_3$ so that $P_1^2 = 6P_2P_3$, and we may put $P_2 = 2a^2c$, $P_3 = 3b^2c$, or $P_2 = a^2c$, $P_3 = 6b^2c$ (a, b, c being arbitrary positive integers) in order to make $6P_2P_3$ a perfect square. By substitution we find $P_1 = 6abc$, and then dropping the common factor c we get two identities as follows:

$$\begin{aligned} &(2a^2)^3 + (3b^2)^3 + (2a^2 + 3b^2 + 6ab)^3 \\ &\qquad = (2a^2 + 3b^2)^3 + (2a^2 + 6ab)^3 + (3b^2 + 6ab)^3 \ , \end{aligned} \tag{11}$$

and

$$\begin{aligned} (a^2)^3 + (6b^2)^3 + (a^2 + 6b^2 + 6ab)^3 &= (a^2 + 6b^2)^3 + (a^2 + 6ab)^3 \\ &\quad + (6b^2 + 6ab)^3 \ . \end{aligned} \tag{12}$$

These two identities provide (9) with two sets of positive integral solutions involving two arbitrary integer parameters a and b. Similarly we can make use of (6) with $m = n = 4$ and $Q = 0$ to obtain infinitely many integral solutions of the equation

$$\sum_{i=1}^{7} X_i^4 = \sum_{i=1}^{7} Y_i^4 \ .$$

In classical number theory

$$\binom{N}{2} = N(N - 1)/2$$

is usually called a "triangular number." It is obvious that not every such number can be expressed as a sum of two triangular numbers. Simple examples $N = 5, 6, 8$ explain this point. These integers are of the form $N \equiv 0, 1, 3 \pmod 5$. Now as an immediate application of (5) we easily show the small

Theorem. Every triangular number $\binom{N}{2}$ with $N \equiv 2, 4 \pmod 5$ can always be expressed as a sum of two triangular numbers.

These numbers may be listed as a sequence:

$$\binom{4}{2},\ \binom{7}{2},\ \binom{9}{2},\ \binom{12}{2},\ \binom{14}{2},\ \binom{17}{2},\ \binom{19}{2},\ \binom{22}{2},\ \binom{24}{2},\ \binom{27}{2},\ \binom{29}{2},\ \cdots$$

In fact, we have explicit relations for $N = 5P + 2$ and $N = 5P - 1$:

$$\binom{5P+2}{2} = \binom{3P+1}{2} + \binom{4P+2}{2}, \qquad \binom{5P-1}{2} = \binom{3P}{2} + \binom{4P-1}{2} .$$

These are easily obtained from (5) by taking $m = n = 2$ and letting $Q = 2P_1 + 1$ or $Q = 2P_1$ in order to delete the two equal terms

$$\binom{Q}{2} = P_1P_2 .$$

These relations may be compared with the formulas

$$\binom{3k+1}{2} + \binom{4k+2}{2} = \binom{5k+2}{2}, \qquad \binom{5k+5}{2} + \binom{12k+10}{2} = \binom{13k+11}{2},$$

and

$$\binom{8k+5}{2} + \binom{15k+10}{2} = \binom{17k+11}{2}, \qquad k = 0, 1, 2, \cdots ,$$

of M. N. Khatri, cited by Sierpiński [6, pp. 84-86]. Sierpiński proves that there exist infinitely many pairs of natural numbers x, y satisfying the system of equations

$$\binom{x+1}{2} + \binom{2y+1}{2} = \binom{3y+1}{2}, \qquad \binom{x+1}{2} - \binom{2y+1}{2} = \binom{y}{2} .$$

Each of these equations is equivalent to the Diophantine equation $x^2 + x = 5y^2 + y$.

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