ARRAYS OF BINOMIAL COEFFICIENTS WHOSE PRODUCTS ARE SQUARES

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1. INTRODUCTION

In [1], Hoggatt and Hansell show that the product of the six binomial coefficients surrounding any particular entry in Pascal's triangle is an integral square. In the preceding article in this Journal [2], Moore generalizes this result by showing that the product of the binomial coefficients forming a regular hexagon with sides on the horizontal rows and main diagonals of Pascal's triangle and having $j + 1$ entries per side is an integral square if j is odd. In the present paper, we derive a fundamental lemma which leads to a generalization of Moore's result and enables us to show that a variety of other interesting configurations of binomial coefficients also yield products which are integral squares.

It will suit our purpose to represent Pascal's triangle (or, more precisely, a portion of it) by a lattice of dots as in Fig. 1. We will have occasion to refer to various polygonal figures and when we do, unless expressly stated to the contrary, we shall always mean a simple closed polygonal curve whose vertices are lattice points. Occasionally, it will be convenient to represent a small portion of Pascal's triangle by letters arranged in the proper position.

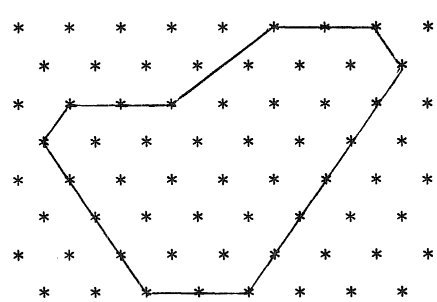


Figure 1

2. THE FUNDAMENTAL LEMMA AND ITS CONSEQUENCES

Lemma 1. The product of the binomial coefficients at the vertices of a pair of parallelograms oriented as in Fig. 2 or Fig. 3 is an integral square. We note that the parallelograms in any pair may overlap and, if they do, the common vertices, if any, must be included twice in the product or, equivalently, must be excluded entirely.

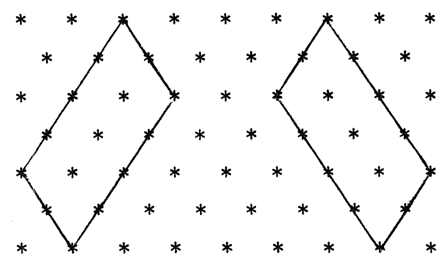


Figure 2

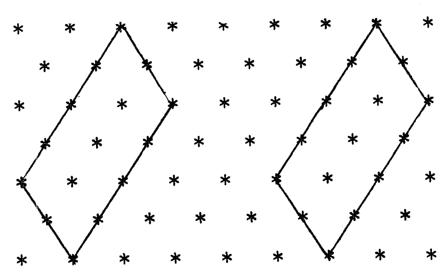


Figure 3

Proof. In the first case, for suitable integers m, n, r, s, and t, the binomial coefficients in question would be

$$\binom{m}{n}, \quad \binom{m+r}{n+r}, \quad \binom{m+s}{n}, \quad \binom{m+s+r}{n+r},$$

$$\binom{m+r}{n+r+t}, \quad \binom{m}{n+r+t}, \quad \binom{m+s+r}{n+s+r+t}, \quad \binom{m+s}{n+s+r+t}.$$

Thus the desired product is

$$\frac{m!}{n!(m-n)!} \cdot \frac{(m+r)!}{(n+r)!(m-n)!} \cdot \frac{(m+s)!}{n!(m-n+s)!}$$

$$\cdot \frac{(m+s+r)!}{(n+r)!(m-n+s)!} \cdot \frac{(m+r)!}{(n+r+t)!(m-n-t)!} \cdot \frac{m!}{(n+r+t)!(m-n-r-t)!}$$

$$\cdot \frac{(m+s+r)!}{(n+s+r+t)!(m-n-t)!} \cdot \frac{(m+s)!}{(n+s+r+t)!(m-n-r-t)!} .$$

This is clearly the square of a rational number. Since it is also an integer, it is an integral square as claimed. The argument for the second case is the same and we omit the details.

As a first consequence of Lemma 1, we now obtain the theorem of Hoggatt and Hansell.

Theorem 2. The product of the six binomial coefficients surrounding $\binom{m}{n}$ in Pascal's triangle is an integral square.

Proof. Let $d = \binom{m}{n}$ and a, b, c, e, f, and g be the six adjacent binomial coefficients as arranged in the array

$$\begin{array}{ccccc} & a & & e & \\ b & & d & & f \\ & c & & g & \end{array} .$$

Since a, b, c, d and e, d, g, f form parallelograms as in Lemma 1, it is immediate that both $abcd^2efg$ and $abcefg$ are integral squares as claimed.

By precisely the same argument, we obtain the following generalization of Theorem 2 which is different from the generalization of Moore mentioned above.

Theorem 3. Let $m > 1$ and $n > 1$ be integers and let H be a convex hexagon whose sides lie on the horizontal rows and main diagonals of Pascal's triangle. Let the numbers of elements on the respective sides of H be m, n, m, n, m, and n in that order, with m being the number of elements along the bottom side. Then the product of the binomial coefficients at the vertices of H is an integral square.

Proof. Of course if $m = n = 2$, this reduces to Theorem 2. In any case, we consider two m by n parallelograms with a common vertex and let a, b, c, d, e, f, and g denote the binomial coefficients at the vertices of the rectangles as indicated in Fig. 4. Clearly, a, b, c, g, f, and e lie at the vertices of a hexagon H of the type described and any such H can be obtained in this way. Therefore, it is again immediate from Lemma 1 that $abcd^2efg$ and $abcefg$ are integral squares.

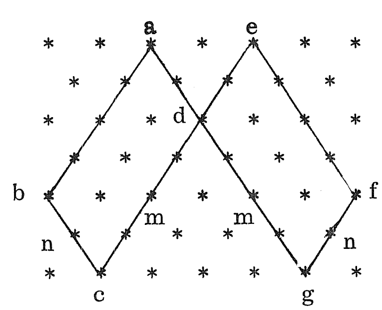


Figure 4

Now let us call the hexagon of Hoggatt and Hansell a fundamental hexagon. Let P be any simple closed polygonal figure. We say that P is tiled with fundamental hexagons if P is "covered" by a set $\mathcal{F}$ of fundamental hexagons in such a way that

(i) the vertices of each F in $\mathcal{F}$ are coefficients in P or in the interior of P,

(ii) each boundary coefficient of P is a vertex of precisely one F in $\mathcal{F}$, and

(iii) each interior coefficient of P is interior to some F in $\mathcal{F}$ or is a vertex shared by precisely two elements of $\mathcal{F}$.

For example, in Fig. 5, G can be tiled by fundamental hexagons and H cannot. Now using the result of Theorem 2 and repeating the essentials of its proof we obtain the following quite general result which leads directly to a generalization of the result of Moore.

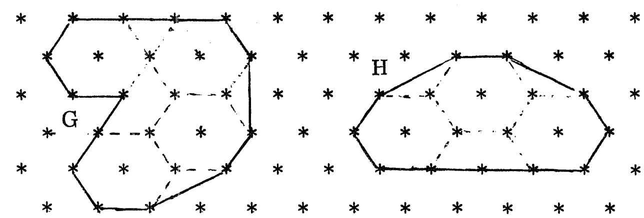


Figure 5

Theorem 4. The product of the binomial coefficients in (the boundary of) any polygonal figure that can be tiled with fundamental hexagons is an integral square.

To see that this generalizes the result of Moore, we prove the following theorem.

Theorem 5. The product of the binomial coefficients in (the boundary of) any convex hexagon with sides oriented along the horizontal rows and main diagonals of Pascal's triangle is an integral square provided the number of coefficients on each side is even.

Proof. In view of Theorem 4, it suffices to show that any hexagon of the type described can be tiled with fundamental hexagons. Let H_n be any such hexagon with n coefficients on its boundary. Plainly, the least possible value of n is 6 which occurs only in the case of a fundamental hexagon. Thus, the result is trivially true in the first possible case. Suppose that it is true for all possible n with $n < k$ where k is any possible value of n with $k > 6$. Since $k > 6$, it follows that at least one side S_1 of H_k must contain at least four coefficients. Without loss of generality, we may presume that S_1 is the lower left-hand side of H_k as indicated in Fig. 6. We may also number the other sides in a counterclockwise direction around H_k. By the induction assumption, it suffices to divide H_k into two hexagons H_i and H_j of the type described and with $i < k$ and $j < k$. We proceed as follows. Let c denote the third coefficient up from the lower end of S_1 and let S be the chord of H_k extending from c and parallel to S_2 as in Fig. 6. Let g be the right-hand end point of S. We distinguish two cases.

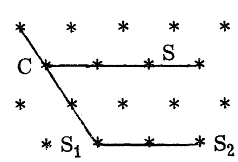


Figure 6

Case 1. If g is on S_3 as in Fig. 7, then the figure a, b, d, h, f, e is an H_i of the desired form since the segment $\overline{dh}$ contains the same number of coefficients as S_2 and the

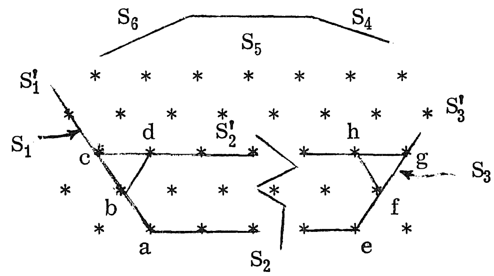


Figure 7

other four sides contain two coefficients each. Also, if we let S_1' denote the upper part of S_1 starting at c, let S_2' denote the line segment $\overline{cg}$, and let S_3' denote the upper part of S_3 starting at g, then S_1' contains two fewer coefficients than S_1, S_2' contains two more coefficients than S_2, and S_3' contains two fewer coefficients than S_3. Thus, the hexagon formed by S_1', S_2', S_3', S_4, S_5, and S_6 is an H_j of the desired type. Finally, since S_2' lies on the interior of H_k (except for its endpoints), it is clear that $i < k$ and $j < k$ as desired.

Case 2. In this case, g lies on S_4 and the appropriate diagram is in Fig. 8. Since the remainder of the argument is essentially the same as for Case 2, we omit the details. This completes the proof.

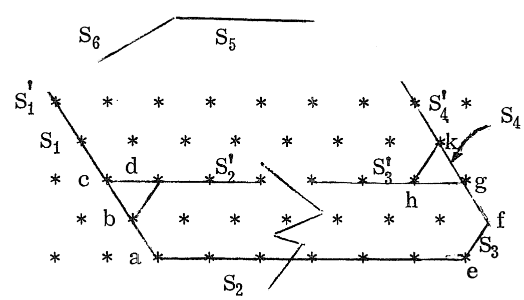


Figure 8

We observe that the convexity conditions of Theorems 3 and 5 are necessary since neither the product of the corner coefficients nor of the boundary coefficients of the hexagon in Fig. 9 is an integral square. Also, it is easy to find examples of convex hexagons where the results of Theorems 3 and 5 do not hold if the condition on the number of elements per side is not met. In fact, we conjecture that the conditions of both theorems are necessary as well as sufficient.

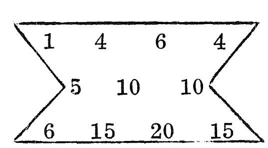


Figure 9

3. SOME ADDITIONAL OBSERVATIONS

In Section 2, we were primarily concerned with hexagons, but it is clear from the fundamental lemma that anything that can be "covered" with pairs of properly oriented parallelograms has the property that the product of those coefficients at the vertices of an odd number of the parallelograms in any such covering is an integral square. Also, if P_1 and P_2 are integral squares which are products of integers and P_3 is the product of those integers common to P_1 and P_2, then P_1P_2/P_3^2 is also an integral square. With these ideas in mind, it is possible to construct an infinite variety of configurations of binomial coefficients whose products are integral squares. The first two examples of such configurations are contained in the following theorems.

Theorem 6. Let K be any convex octagon with sides oriented along the horizontal and vertical rows and main diagonals of Pascal's triangle. Let the number of vertices on the various sides be 2r, 2s, t, 2u, 2v, 2u, t, and 2s as indicated in Fig. 10 where r, s, t, u, and v are positive integers. Then the product of the boundary coefficients is an integral square.

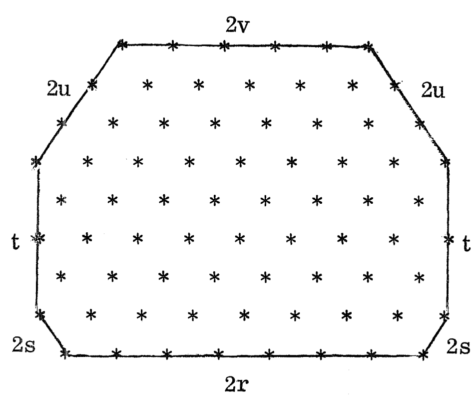


Figure 10

Proof. The proof of this theorem is essentially the same as for Theorem 5 and will be omitted.

In Theorem 6, the convexity condition is not necessary, but it is not presently clear how the theorem should read if this condition is removed. While the octagons of Theorem 6 can be tiled with fundamental hexagons, the octagon of Fig. 11 cannot. It can, however, be tiled with pairs of properly oriented parallelograms (or a combination of parallelograms and fundamental hexagons, if you prefer) and it follows from the fundamental lemma that the product of the boundary coefficients is an integral square.

Also note that the products of the corner coefficients in Fig. 10 of Theorem 6 and in Fig. 11 need not be squares. However, as the following theorem shows, at least one class of octagons exists for which the product of the corner coefficients is always an integral square.

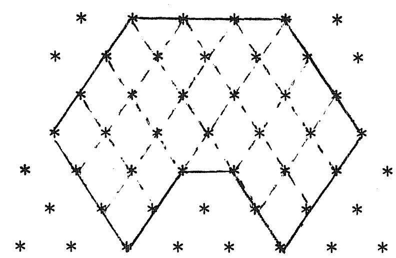


Figure 11

Theorem 7. Let K be a convex octagon formed as in Fig. 12 by adjoining parallelograms with r and s and r and t elements on a side to a parallelogram with r elements on each side. Then the product of the corner coefficients of the octagon is an integral square.

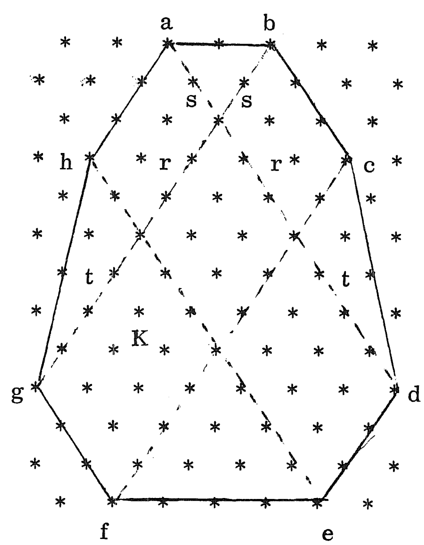


Figure 12

Proof. Let a, b, c, d, e, f, g, and h denote the corner coefficients of the octagon as indicated in Fig. 12. Since a, d, e, and h and b, c, f, and g are the vertices of rectangles oriented as in the fundamental lemma, it is clear that their product is an integral square as claimed.

Again it is clear that the convexity condition of Theorem 7 is not necessary. The most general statement which we can make at the present time is that the product of the corner coefficients of any octagon formed by joining (as in Fig. 13) the vertices of pairs of parallelograms oriented as in the fundamental lemma is an integral square. It is not clear that even this condition is necessary. See Usiskin [3].

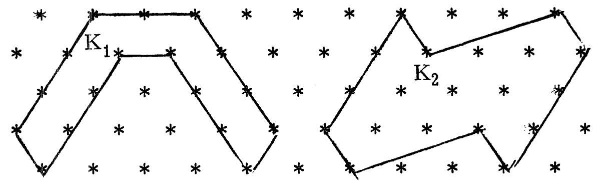
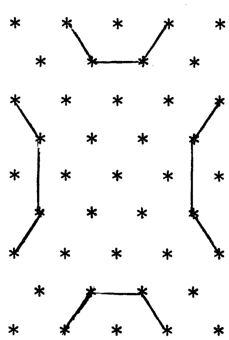


Figure 13

We now give, without proof, several examples of configurations of binomial coefficients whose produces are integral squares. Each example given is a (sometimes not simple, closed, or connected) polygon and it is intended that one consider the product of the boundary coefficients only. Note that it is quite possible to find solid and other non-polygonal arrays whose products are integral squares



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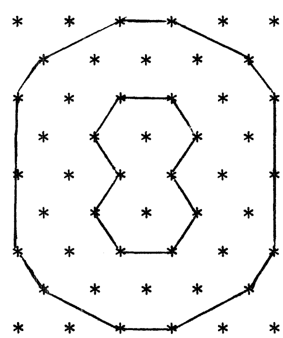

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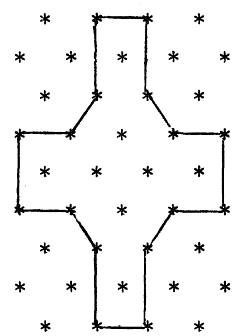

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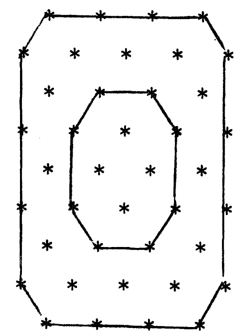

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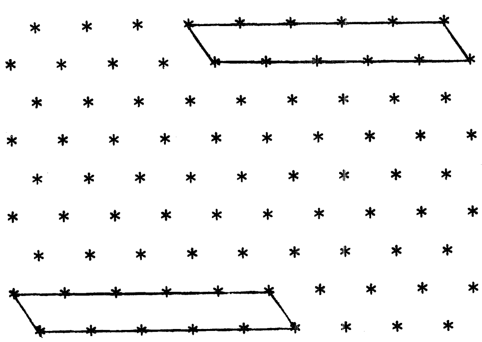

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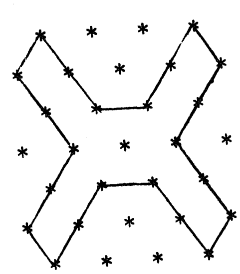

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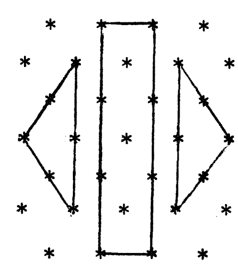

VII

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2. C. Moore, "More Hidden Hexagon Squares," The Fibonacci Quarterly,
3. Zalman Usiskin, "Perfect Square Patterns in the Pascal Triangle," Math. Mag., 46 (1973), No. 4, pp. 203-207.