A FIBONACCI PROBABILITY FUNCTION

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1. THE FIBONACCI DISTRIBUTION

Consider the following Markov Process. To begin, a marker is placed in slot number zero. At each minute thereafter, a coin is flipped. If it comes up heads, the marker is moved up one slot. If it comes up tails, the marker is moved back to position zero. Let X_n be the number of flips needed to advance the marker to position n. We would like to investigate the distribution of the random variable, X_n. For the case $n = 1$, the random variable is simply geometric (i.e., X_1 = number of trials until the first success occurs). Let us therefore, start with the case $n = 2$ and probability of a head, $p = 1/2$.

Let $P(X_2 = k) = p_2(k)$, $k = 2, 3, 4, \cdots$. Now,

$$p_2(2) = P(HH) = 1/2^2, \quad p_2(3) = P(THH) = 1/2^3$$

and

$$\begin{aligned} p_2(k+3) &= P(k \text{ trials with no run of two heads}) \cdot P(THH) \\ &= (A_{2,k}/2^k) \cdot (1/2^3) = A_{2,k}/2^{k+3}, \quad k = 1, 2, 3, \cdots, \end{aligned} \tag{1}$$

where $A_{2,k}$ = number of arrangements of k heads and tails with no two consecutive heads. In order to evaluate $A_{2,k}$, we note that we may classify the allowable arrangements according to whether the last tail is in the k^{th} or $(k-1)^{st}$ position. Letting $a_{2,k,i}$ = number of arrangements of k heads and tails having no two consecutive heads and having a tail in the i^{th} position, $i = k, k-1$, gives $A_{2,k} = a_{2,k,k} + a_{2,k,k-1}$. But, $a_{2,k,k} = A_{2,k-1}$ and $a_{2,k,k-1} = A_{2,k-2}$, yielding

$$A_{2,k} = A_{2,k-1} + A_{2,k-2}\,. \tag{2}$$

For $k = 1$, the possible arrangements are simply H and T. Thus, $A_{2,1} = 2$. For $k = 2$, the possible arrangements are HT, TH, TT. Thus, $A_{2,2} = 3$. Combining (1), (2) and the preceding, we have

$$p_2(k) = F_{k-2}/2^k \qquad k = 2, 3, 4, \cdots, \tag{3}$$

where $F_k = k^{th}$ Fibonacci number (with $F_0 = F_1 = 1$). Certainly, a good name for this is the Fibonacci Probability Distribution.

The cumulative distribution function of X_2 is given by

$$G_2(x) = P(X_2 \le x) = \sum_{k=2}^{[x]} F_{k-2}/2^k \quad \text{for } x \ge 2, \text{ and zero otherwise,} \tag{4}$$

where $[x]$ = largest integer less than or equal to x. In order to close this sum, we simply note the following

Lemma.

$$\sum_{j=0}^{n} 2^{n-j}F_j = 2^{n+2} - F_{n+3} .$$

Proof. By induction, if $n = 0$, the left-hand side is simply $F_0 = 1$ and the right-hand side is $2^2 - F_3 = 4 - 3 = 1$. Now assuming the result for n, consider

$$\sum_{j=0}^{n+1} 2^{n+1-j}F_j = F_{n+1} + 2\sum_{j=0}^{n} 2^{n-j}F_j = F_{n+1} + 2(2^{n+2} - F_{n+3})$$

$$= F_{n+1} + 2^{n+3} - 2F_{n+3} = 2^{n+3} - (F_{n+3} + F_{n+3} - F_{n+1})$$

$$= 2^{n+3} - F_{n+4} = 2^{(n+1)+2} - F_{(n+1)+3} \quad \text{q. e. d.}$$

Applying this Lemma, we see that $G_2(x)$ is simply

(5) $$G_2(x) = \sum_{k=2}^{[x]} F_{k-2}/2^k = 2^{-[x]}\sum_{k=0}^{[x]-2} 2^{[x]-2-k}F_k = 2^{-[x]}(2^{[x]} - F_{[x]+1}).$$

So,

(6) $$G_2(x) = 1 - F_{[x]+1}/2^{[x]} \quad \text{if } x \geq 0 \text{ and } 0 \text{ otherwise.}$$

The factorial moment generating function $M_2(t) = Et^{X_2}$, is easily obtained,

$$M_2(t) = \sum_{k=2}^{\infty} t^k p_2(k) = \sum_{k=2}^{\infty} t^k F_{k-2}/2^k = (t/2)^2 \sum_{k=0}^{\infty} F_k(t/2)^k = \tfrac{1}{4}t^2 g(\tfrac{1}{2}t) ,$$

where

$$g(x) = \sum_{k=0}^{\infty} F_k x^k ,$$

the generating function for the Fibonacci numbers, that is, $g(x) = (1 - x - x^2)^{-1}$. Therefore,

(7) $$M_2(t) = t^2/(4 - 2t - t^2)$$

and

$$\frac{d^m}{dt^m} M_2(t)\bigg|_{t=1} = EX_2(X_2 - 1)\cdots(X_2 - m + 1) = f_{2,m} ,$$

the m^{th} factorial moment. The usual moment generating function is, of course, $M_2(e^t)$. Making the substitution, $u = t - 1$, we produce $m_2(u) = M_2(u + 1)$, for which $m_2^{(m)}(0) = f_{2,m}$. A partial fraction decomposition of the preceding yields

$$m_2(u) = -1 + \frac{2u - 2}{u^2 + 4u - 1} = -1 + \left(\frac{5 + 3\sqrt{5}}{5}\right)\left(\frac{1}{u + \alpha}\right) + \left(\frac{5 - 3\sqrt{5}}{5}\right)\left(\frac{1}{u + \beta}\right) ,$$

where $\alpha = 2 + \sqrt{5}$ and $\beta = 2 - \sqrt{5}$. Expanding both fractions as power series, elementary computations yield

$$m_2(u) = -1 + \sum_{j=0}^{\infty} \left[\frac{3(\alpha^j + \beta^j) + (\alpha^{j+1} + \beta^{j+1})}{5}\right] u^j . \tag{8}$$

Since the coefficient of u^j is $m^{(j)}(0)/j!$, comparing terms in (8), we have

$$f_{2,m} = m!\,[3(\beta^j + \alpha^j) + (\beta^{j+1} + \alpha^{j+1})]/5 . \tag{9}$$

2. THE POLY-NACCI DISTRIBUTION

Let us now proceed along the lines of section one, to develop the situation for the case of n greater than or equal to two. Let $P(X_n = k) = p_n(k)$ $k = n, n + 1, \cdots$. Here we have $p_n(n) = P$ (n heads in a row) $= (1/2)^n$, $p_n(n + 1) = P$ (one tail followed by n consecutive heads) $= (1/2)^{n+1}$ and

$$\begin{aligned} p_n(k + n + 1) &= P(k \text{ trials with no run of } n \text{ heads}) \cdot p_n(n + 1) \\ &= (A_{n,k}/2^k)\cdot(\tfrac{1}{2})^{n+1} = A_{n,k}/2^{n+k+1} \quad k = 1, 2, 3, \cdots, \end{aligned}$$

where $A_{n,k}$ = number of arrangements of heads and tails with no run of n heads. Again, we may evaluate $A_{n,k}$ by letting $a_{n,k,i}$ = number of arrangements of k heads and tails having no run of n heads and the last tail in the i^{th} position, $i = k, k - 1, \cdots, k - n + 1$. Thus,

$$\sum_{j=0}^{n-1} a_{n,k,k-j} ,$$

but

$$a_{n,k,k-j} = A_{n,k-(j+1)} \qquad j = 0, 1, \cdots, n - 1 .$$

So, analogously to (2),

$$A_{n,k} = \sum_{j=0}^{n-1} A_{n,k-(j+1)} \quad \text{where} \quad A_{n,i} = 2^i \quad 1 = 0, 1, 2, \cdots, n - 1 . \tag{10}$$

At this point, it is convenient to define the k^{th} poly-nacci number of order n, $F_{n,k}$, by the recurrence

(11) $$F_{n,k} = F_{n,k-1} + F_{n,k-2} + \cdots + F_{n,k-n} \qquad k = 1, 2, 3, \cdots ,$$

where $F_{n,0} = 1$ and $F_{n,-r} = 0$.

Using this notation, we may write

(12) $$p_n(k) = F_{n,k-n}/2^k \qquad k = n, n+1, n+2, \cdots .$$

The cumulative distribution function

(13) $$G_n(x) = P(X_n \le x) = \sum_{k=n}^{[x]} F_{n,k-n}/2^k \qquad \text{for } x \ge n .$$

As in Section One, we state

Lemma.

$$\sum_{j=0}^{N} 2^{N-j} F_{n,j} = 2^{N+n} - F_{n,N+n+1} .$$

Proof. By induction on N, when $N = 0$, the left-hand side is simply $F_{n,0} = 1$. The right-hand side is $2^n - F_{n,n+1}$, but

$$F_{n,n+1} = \sum_{k=1}^{n} F_{n,k} = \sum_{k=0}^{n-1} 2^k = 2^n - 1 ,$$

establishing the result for $N = 0$. Assuming the result for N, let us consider

$$\sum_{j=0}^{N+1} 2^{N+1-j} F_{n,j} = F_{n,N+1} + 2\sum_{j=0}^{N} 2^{N-j} F_{n,j} = F_{n,N+1} + 2^{N+n+1} - 2F_{n,N+n+1}$$

$$= 2^{n+(N+1)} - (F_{n,N+n+1} + F_{n,N+n+1} - F_{n,N+1}) .$$

Since

$$F_{n,N+n+1} = \sum_{j=0}^{n-1} F_{n,N+1+j}, \qquad F_{n,N+n+1} - F_{n,N+1} = \sum_{j=1}^{n-1} F_{n,N+1+j} ,$$

and

$$\sum_{j=0}^{N+1} 2^{N+1-j} F_{n,j} = 2^{n+(N+1)} - \left(F_{n,N+1+n} + \sum_{j=1}^{n-1} F_{n,N+1+j}\right)$$

$$= 2^{n+(N+1)} - F_{n,N+1+n+1} = 2^{n+(N+1)} - F_{n,n+(N+1)+1} , \quad \text{q.e.d.}$$

Applying the Lemma,

$$G_n(x) = \sum_{k=n}^{[x]} F_{n,k-n}/2^k = 2^{-[x]} \sum_{k=n}^{[x]} 2^{[x]-k} F_{n,k-n}$$

$$= 2^{-[x]} \sum_{r=0}^{[x]-n} 2^{[x]-n-r} F_{n,r} = 2^{-[x]}(2^{[x]} - F_{n,[x]+1}) .$$

Thus, Eq. (13) reduces to a form almost identical to (6), namely,

$$G_n(x) = 1 - 2^{-[x]} F_{n,[x]+1} \quad \text{if } x \geq n \text{ and } 0 \text{ otherwise.} \tag{14}$$

Finally, the factorial moment generating function

$$M_n(t) = Et^{X_n} = \sum_{k=n}^{\infty} t^k p_n(k) = \sum_{k=n}^{\infty} t^k F_{n,k-n}/2^k = (\tfrac{1}{2}t)^n \sum_{k=0}^{\infty} F_{n,k}(\tfrac{1}{2}t)^k$$

$$= (\tfrac{1}{2}t)^n g_n(\tfrac{1}{2}t) ,$$

where

$$g_n(x) = \sum_{k=0}^{\infty} F_{n,k} x^k ,$$

the generating function for the n^{th} order poly-nacci numbers. Since $g_n(x)$ is easily seen to be $g_n(x) = (1 - x - x^2 - \cdots - x^n) = (1 - x)/(1 - 2x + x^{n+1})$, we obtain

$$M_n(t) = t^n/(2^n - 2^{n-1}t - \cdots - t^n) = t^n(2 - t)/(2^{n+1}(1 - t) + t^{n+1}) . \tag{15}$$

Unfortunately, a closed form expression for the $f_{n,m}$ the m^{th} factorial moment of X_n, is not readily available.

3. THE GENERALIZED POLY-NACCI DISTRIBUTION

Let us briefly apply the methods of Section 2 to the case where the probability of a head is p, $0 < p < 1$, and not necessarily $1/2$. Let $q = 1 - p$ and as before let X_n = number of trials needed to reach position n. Letting $p_n(k) = P(X_n = k)$, $p_n(n) = p^n$, $p_n(n+1) = qp^n$, $p_n(n + j + 1) = qp^n p_{n,j}$ $j = 1, 2, 3, \cdots$, where $p_{n,j} = P$(j trials with no run of n heads). Now, $p_{n,j} = 1$ for $j = 0, 1, 2, 3, \cdots, n - 1$ and breaking down the probability according to the number of the last tail, we obtain

$$p_{n,j} = \sum_{r=1}^{n} qp^{r-1} p_{n,j-r} , \qquad j = n, n+1, n+2, \cdots .$$

Thus, if we define

$$F_p(n;j) = q \sum_{r=1}^{n} p^{r-1} F_p(n; j - r) \qquad j = 0, 1, 2, \cdots$$

with $F_p(n;0) = 1$ and $F_p(n;-k) = 0$, we may write $p_n(k) = p^n F_p(n;k-n)$, $k = n, n+1, \cdots$. The $F_p(n;j)$ being the "Poly-nacci Polynomials of order n in p." For example, the first few Fibonacci Polynomials $F_x(2;j)$ are given by: 1, $1 - x$, $1 - x$, $(1-x)^2(1+x)$, $(1-x)^3(1+x) + (1-x)^2 x$, $\cdots$. The cumulative distribution function of X_n is

$$G_n(x) = \sum_{k=n}^{[x]} p_n(k) = \sum_{k=n}^{[x]} p^n F_p(n; k - n) \qquad \text{for} \quad x \geq n .$$

That is,

$$G_n(x) = p^n \sum_{i=0}^{[x]-n} F_p(n; i) .$$

It is easy to show by induction that

$$\sum_{i=0}^{M} F_p(n;i) = (q - F_p(n, M + n + 1))/qp^n$$

so that

(16) $$G_n(x) = 1 - q^{-1} F_p(n; [x] + 1) \quad \text{if} \quad x \geq n \text{ and } 0 \text{ otherwise.}$$

The generating function for the $F_p(n;i)$,

$$g_n(x;p) = \sum_{i=0}^{\infty} F_p(n;i)x^i = \left[1 - qx \sum_{j=0}^{n-1} (px)^j\right]^{-1} = (1 - px)/(1 - x + qp^n x^{n+1}) .$$

Thus, the factorial moment generating function for X is

(17) $$M_n(t;p) = \sum_{k=n}^{\infty} t^k p_n(k) = p^n t^n \sum_{i=0}^{\infty} F_p(n;i)\, t^i = p^n t^n (1 - pt)/(1 - t + qp^n t^{n+1}).$$

So, for instance,

$$\frac{d}{dt} M_n(t;p)\bigg|_{t=1} = EX_n = (1 - p^n)/qp^n ,$$

which for $p = 1/2$ yields $E_{1/2}X_n = 2^{n+1} - 2$. Of course, results concerning the mean are easily obtained by developing the recurrence for EX_{n+1} in terms of EX_n but the same is not true for the higher moments. Lastly, the analysis of the probabilistic situations such as the preceding may well reveal insights into the Fibonacci numbers and their extensions.