GENERALIZED HIDDEN HEXAGON SQUARES

A. K. GUPTA
The University of Michigan, Ann Arbor, Michigan 48104

The triangular array of binomial coefficients is well known. Recently, Hoggatt and Hansel [2] have obtained a very surprising result involving these numbers. Stanton and Cowan [3] and Gupta [1] have generalized this triangular array to a tableau. In this paper, we generalize the results due to Hoggatt and Hansel.

Let, for any positive integer m and any integer n, $\binom{m}{n} = 0$ if either $n > m$ or $n < 0$. Then we prove the following theorem.

Theorem. The product of the six binomial coefficients spaced around $\binom{m}{n}$, viz.,

$$\binom{m - r_1}{n - r_2}\binom{m - r_1}{n}\binom{m}{n - r_2}\binom{m + r_2}{n + r_1}\binom{m + r_2}{n}\binom{m}{n + r_1} ,$$

where r_1 and r_2 are positive integers, is a perfect integer square.

Proof. The product of the six binomial coefficients is

$$\frac{(m - r_1)!}{(n - r_2)!(m - r_1 - n + r_2)!} \cdot \frac{(m - r_1)!}{(n)!(m - r_1 - n)!} \cdot \frac{(m)!}{(m - r_2)!(m - n + r_2)!} \cdots$$

$$= \left[\frac{(m - r_1)!(m)!(m + r_2)!}{(n - r_2)!(m - r_1 - n + r_2)!(n)!(m - r_1 - n)!(m - n + r_2)!(n + r_1)!}\right]^2 .$$

Now, the product of binomial coefficients is an integer, since each binomial coefficient is an integer. And the square of a rational number is an integer if and only if the rational number is an integer. Hence the product is an integer square.

It is interesting to note that

$$\binom{m}{n - r_2}\binom{m - r_1}{n}\binom{m + r_2}{n + r_1} = \binom{m - r_1}{n - r_2}\binom{m + r_2}{n}\binom{m}{n + r_1} ,$$

which is what really happens to make the product of six numbers a perfect square.

Corollary 1. If $r_1 = r_2$, we get the product of six binomial coefficients which are equally spaced around $\binom{m}{n}$.

Corollary 2. If $r_1 = r_2 = 1$, we get the product of six binomial coefficients that surround $\binom{m}{n}$. This is the result of Hoggatt and Hansel [2]. Hence their result is a very special case of our general theorem.

By taking different values for r_1 and r_2, we can obtain several configurations which yield products of binomial coefficients which are squares. In fact, one can build up a long serpentine configuration, or snowflake curves, as noted by Hoggatt and Hansel.

Note that the theorem holds for generalized binomial coefficients (and hence for q-binomials), and in particular for the Fibonomial coefficients.

REFERENCES

1. A. K. Gupta, "On a 'Square' Functional Equation," unpublished.
2. V. E. Hoggatt, Jr., and Walter Hansel, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2 (April, 1971), pp. 120 and 133.
3. R. G. Stanton and D. D. Cowan, "Note on a 'Square' Functional Equation," Siam Review, Vol. 12 (1970), pp. 277-279.

LETTER TO THE EDITOR

Dear Editor:

Here are two related problems for the Fibonacci Quarterly, based on some remarkable things discovered last week by Ellen Crawford (a student of mine).

Problem 1. Prove that if m and n are any positive integers, there exists a solution x to the congruence

$$F_x \equiv m \pmod{3^n} .$$

Solution. Let m be fixed: we shall show that it is possible to solve the simultaneous congruences

$$(*) \qquad \begin{aligned} F_x &\equiv m \pmod{3^n} \\ F_x + F_{x+1} &\not\equiv 0 \pmod{3} . \end{aligned}$$

This is clearly true for n = 1. It is also easy to prove by induction, using

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1} ,$$

(Continued on page 79.)