SETS OF BINOMIAL COEFFICIENTS WITH EQUAL PRODUCTS

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1. INTRODUCTION

In [2], Hoggatt and Hansell show that the product of the six binomial coefficients surrounding any particular entry in Pascal's triangle is an integral square. They also observe that the two products of the alternate triads of these six numbers are equal. Quite remarkably, Gould conjectured and Hillman and Hoggatt [1] have now proved that the two greatest common divisors of the numbers in the above-mentioned triads are also equal though their least common multiples are, in general, not equal. Hillman and Hoggatt also generalize the greatest common divisor property to more general arrays.

The integral square property was further investigated by Moore [4], who showed that the result is true for any regular hexagon of binomial coefficients if the number of entries per side is even, and by the present author [3], who generalized the earlier results to non-regular hexagons, octagons, and other arrays of binomial coefficients whose products are squares.

In the present paper, we generalize the equal product property of Hoggatt and Hansell along the lines of [3] and also make some observations and conjectures regarding a generalized greatest common divisor property.

It will suit our purpose to represent Pascal's triangle (or, more precisely, a portion of it) by a lattice of dots as in Fig. 1. We will have occasion to refer to various polygonal figures and when we do, unless expressly stated to the contrary, we shall always mean a simple closed polygonal curve whose vertices are lattice points. Occasionally, it will be convenient to represent a small portion of Pascal's triangle by letter arranged in the proper position.

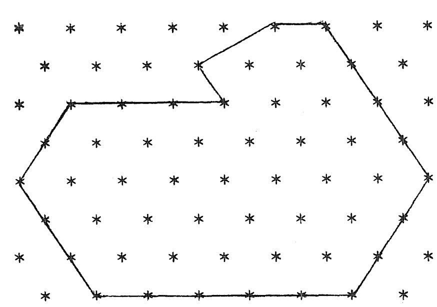


Fig. 1

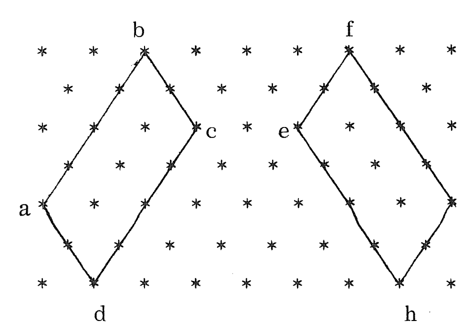


Fig. 2

2. SETS OF BINOMIAL COEFFICIENTS WITH EQUAL PRODUCTS

As in [3], we begin by deriving a fundamental lemma which is basic to all of the other results of this section.

Lemma 1. Consider two parallelograms of binomial coefficients oriented as in Fig. 2 and with corner coefficients a, b, c, d and e, f, g, h as indicated. Then the products acfh and bdeg are equal.

Proof. For suitable integers m, n, r, s, and t, the binomial coefficients in question may be represented in the form

$$a = \binom{m+s}{n}, \quad b = \binom{m}{n}, \quad c = \binom{m+r}{n+r}, \quad d = \binom{m+s+r}{n+r},$$

$$c = \binom{m+r}{n+r+t}, \quad f = \binom{m}{n+r+t}, \quad g = \binom{m+s}{n+s+r+t}, \quad h = \binom{m+s+r}{n+s+r+t}.$$

Thus, the desired products are

$$\begin{aligned} \text{acfh} &= \frac{(m+s)!}{n!\,(m-n+s)!} \cdot \frac{(m+r)!}{(n+r)!\,(m-n)!} \\ &\quad \cdot \frac{m!}{(n+r+t)!\,(m-n-r-t)!} \cdot \frac{(m+s+r)!}{(n+s+r+t)!\,(m-n-t)!} \end{aligned}$$

and

$$\begin{aligned} \text{bdeg} &= \frac{m!}{n!\,(m-n)!} \cdot \frac{(m+s+r)!}{(n+r)(m-n+s)!} \\ &\quad \cdot \frac{(m+r)}{(n+r+t)!\,(m-n-t)!} \cdot \frac{(m+s)!}{(n+r+s+t)!\,(m-n-r-t)!} \end{aligned}$$

and these are clearly equal as claimed.

As a first consequence of Lemma 1, we obtain the equal product result of Hoggatt and Hansell.

Theorem 2. Let a, b, c, d, e, f, and g denote binomial coefficients as in the array

$$\begin{array}{ccccc} & a & & b & \\ f & & g & & c \\ & e & & d & \end{array} .$$

Then aec = fbd.

Proof. The parallelograms a, f, e, g and b, g, d, c are oriented as in Lemma 1. Therefore, fgbd = aegc and this implies the desired result.

By essentially the same argument, we obtain the following more general statement about products of coefficients at the vertices of hexagons in Pascal's triangle.

Theorem 3. Let $m > 1$ and $n > 1$ be integers and let H be a convex hexagon whose sides lie on the horizontal rows and main diagonals of Pascal's triangle. Let the number of coefficients on the respective sides of H be m, n, m, n, m, and n in that order, and let a, b, c, d, e, and f be the coefficients in cyclic order at the vertices of H. Then ace = bdf.

<u>Proof</u>. Without loss in generality, we may take m to be the number of coefficients along the bottom side of H. If we consider two m-by-n parallelograms with a common vertex and with corner coefficients a, b, c, d, e, f, and g as in Fig. 3, then, again by Lemma 1, fgbd = aegc and this implies the equality claimed.

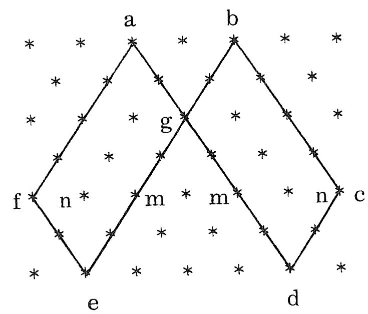


Fig. 3

Now, as in [3], let us call the hexagons of Hoggatt and Hansell fundamental hexagons and say that a polygonal figure P on Pascal's triangle is <u>tiled</u> with fundamental hexagons if P is "covered" by a set $\mathcal{F}$ of fundamental hexagons F in such a way that

i. The vertices of each F in $\mathcal{F}$ are coefficients in P or in the interior of P.
ii. Each boundary coefficient of P is a vertex of precisely one F in $\mathcal{F}$, and
iii. Each interior coefficient of P is interior to some F in $\mathcal{F}$ or is a vertex shaped by precisely two elements of $\mathcal{F}$.

We can then prove the following result.

<u>Theorem 4.</u> Let P_n be a polygonal figure on Pascal's triangle with boundary coefficients $a_1, a_2, \cdots, a_n$ in order around P_n. If P_n can be tiled by fundamental hexagons, then $n = 2s$ for some $s \geq 3$ and

$$\prod_{i=1}^{s} a_{2i-1} = \prod_{i=1}^{s} a_{2i} .$$

<u>Proof</u>. Suppose that P_n can be tiled with r fundamental hexagons. The proof proceeds by induction on r. Clearly the least value of r is 1 which occurs only in the case of the fundamental hexagon itself. In this case, n = 6 and the result is true by Theorem 2. Now suppose that the result is true for any polygon that can be tiled with fewer than k fundamental hexagons where $k > 1$ is fixed and let P_n be a polygon that can be tiled with k fundamental hexagons. Let H be one of the hexagons which tiles P_n and contains at least one boundary point of P_n. We distinguish five cases.

Case 1. H contains just one boundary point of P_n. Without loss in generality, we may let a_n be the boundary point of P_n which is in H. Let a'_n, a'_{n+1}, a'_{n+2}, a'_{n+3}, and a'_{n+4} denote the other five boundary points of H in order around H. Let P_m be the polygon obtained from P_n by deleting H. Then the boundary points of P_m are $a_1, a_2, \cdots, a_{n-1}, a'_n, a'_{n+1}, a'_{n+2}, a'_{n+3}$, and a'_{n+4}. Thus, $m = n + 4$. Also, since P_m can be tiled by $k - 1$ fundamental hexagons, $n + 4 = m = 2t$ for some t and

$$a_1 a_3 \cdots a_{n-1} a'_{n+1} a'_{n+3} = a_2 a_4 \cdots a_{n-2} a'_n a'_{n+2} a'_{n+4} .$$

But, since H is a fundamental hexagon, it follows that

$$a_n a'_{n+1} a'_{n+3} = a'_n a'_{n+2} a'_{n+4}$$

and this clearly implies that

$$\prod_{i=1}^{s} a_{2i-1} = \prod_{i=1}^{s} a_{2i}$$

since $n = 2t - 4 = 2s$. This completes the proof for Case 1.

Cases 2-4. In these cases, respectively, H contains 2, 3, 4, or 5 boundary points of P_n. We omit the proofs of these cases since they essentially duplicate the proof of Case 1. This completes the proof.

With Theorem 4 and Lemma 1 as our principal tools we are now able to give several quick results.

Theorem 5. Let H_n be a convex hexagon with an even number of coefficients per side, with sides oriented along the horizontal rows and main diagonals of Pascal's triangle, and with boundary coefficients $a_1, a_2, \cdots, a_n$ in order around H_n. Then $n = 2s$ for some $s \geq 3$ and

$$\prod_{i=1}^{s} a_{2i-1} = \prod_{i=1}^{s} a_{2i} .$$

Proof. This is an immediate consequence of Theorem 4 since H_n can be tiled by fundamental hexagons as shown in Theorem 5 of [3].

Theorem 6. Let K_n be any convex octagon with sides oriented along the horizontal and vertical rows and main diagonals of Pascal's triangle and with boundary coefficients $a_1, a_2, \cdots, a_n$ in order around K_n. Let the number of coefficients on the various sides of K_n be $2r$, $2s$, t, $2u$, $2v$, t, and $2s$ as indicated in Fig. 4. Then $n = 2h$ for some $h \geq 4$ and

$$\prod_{i=1}^{s} a_{2i-1} = \prod_{i=1}^{s} a_{2i} .$$

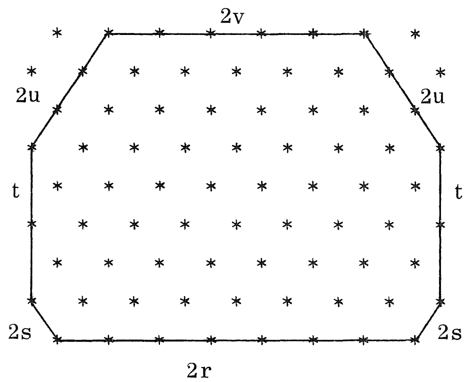


Fig. 4

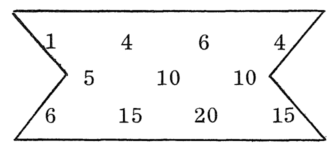


Fig. 5

Proof. The proof is the same as for Theorem 5 and will be omitted.

We observe that the convexity conditions in both Theorems 3 and 5 are necessary since neither result is true for the hexagon of Fig. 5. Also, it is easy to find examples of convex hexagons where the results of Theorems 3 and 5 do not hold if the conditions on the number of elements per side are not met. In fact, we conjecture that the conditions in these theorems are both necessary and sufficient. On the other hand, the convexity condition of Theorem 6 is not necessary since the result holds for the octagon of Fig. 6 which is clearly not convex. We make no conjecture regarding necessary and sufficient conditions for the result of Theorem 6 to hold for octagons in general, or indeed, for hexagons whose sides may not lie along the horizontal rows and main diagonals of Pascal's triangle. We note that the octagon of Fig. 6 cannot be tiled by fundamental hexagons but can be tiled by pairs of properly oriented "fundamental parallelograms" as indicated by the shading in the figure. Thus, the most general theorem for these and other polygons will most likely have to be couched in terms of tilings by sets of pairs of fundamental parallelograms.

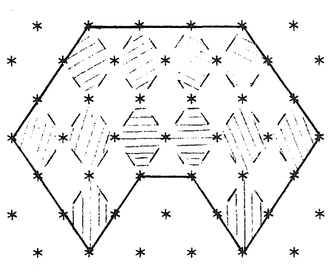


Fig. 6

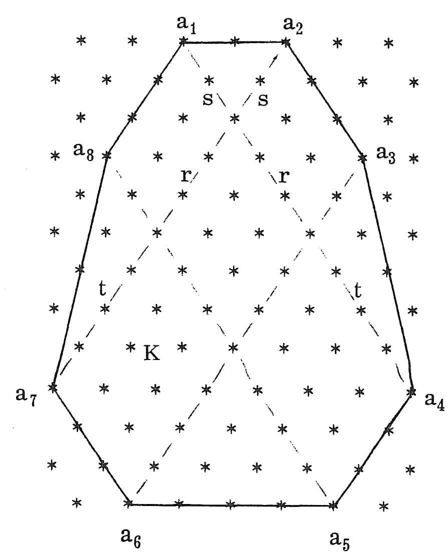


Fig. 7

3. ADDITIONAL COMMENTS ON EQUAL PRODUCTS

Theorem 3 gives an equal product result for the corner coefficients of hexagons and it is natural to seek similar results for octagons. It is easy to find octagons like those in Figs. 4 and 6 for which the equal product property on vertices does not hold. Nevertheless, it is possible to find classes of octagons for which the equal product property does hold for the products of alternate corner coefficients.

Theorem 7. Let K be a convex octagon formed as in Fig. 7 by adjoining parallelograms with r and s and r and t elements on a side to a parallelogram with r elements on each side. If the corner coefficients are $a_1, a_2, \cdots, a_8$ as shown, then

$$a_1 a_3 a_5 a_7 = a_2 a_4 a_6 a_8 .$$

Proof. We have only to observe that a_1, a_4, a_5, a_8 and a_2, a_3, a_6, a_7 are vertices of pairs of parallelograms oriented as in Lemma 1. The result is then immediate.

Again it is clear that the convexity condition of Theorem 7 is not necessary. The proof, after all, rests on the presence of the properly oriented pairs of parallelograms. In precisely the same way we show that $a_1 a_3 a_5 a_7 = a_2 a_4 a_6 a_8$ for each of the three octagons of Fig. 8. Note that for K_2 the two products are not products of alternate vertices around the octagon.

Clearly the preceding methods can be used to obtain a wide variety of configurations of binomial coefficients which divide into sets with equal products. As illustrations we give several examples of polygons (sometimes not closed, simple, or connected) with this property.

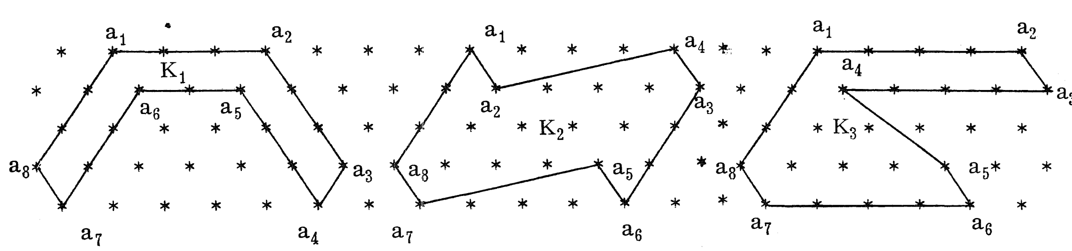


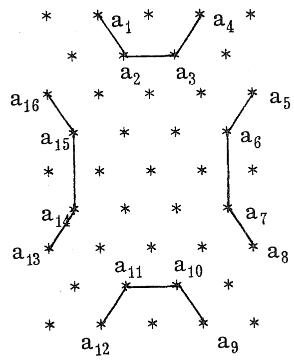
Fig. 8

4. THE GREATEST COMMON DIVISOR PROPERTY

As mentioned in Section 1, if the array

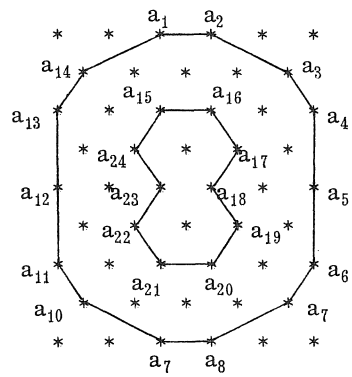
$$\begin{array}{ccccc} & a & & b & \\ c & & d & & e \\ & f & & g & \end{array}$$

represents coefficients from Pascal's triangle, then $afe = cbg$ and Hillman and Hoggatt



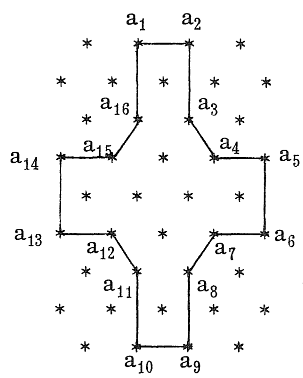
$$\prod_{i=1}^{8} a_{2i-1} = \prod_{i=1}^{8} a_{2i}$$

I



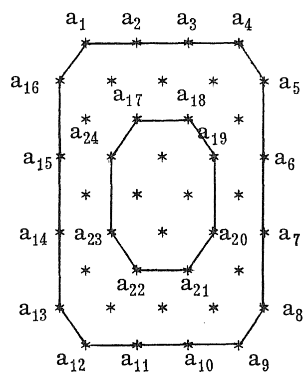
$$\prod_{i=1}^{12} a_{2i-1} = \prod_{i=1}^{12} a_{2i}$$

II



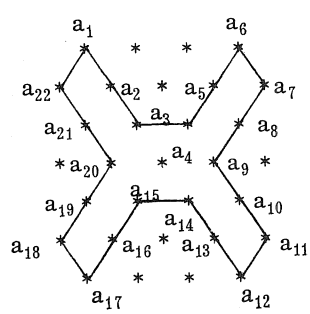
$$\prod_{i=1}^{8} a_{2i-1} = \prod_{i=1}^{8} a_{2i}$$

III



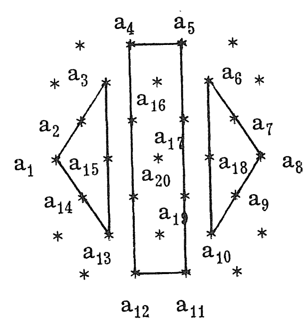
$$\prod_{i=1}^{12} a_{2i-1} = \prod_{i=1}^{12} a_{2i}$$

IV



$$\prod_{i=1}^{11} a_{2i-1} = \prod_{i=1}^{11} a_{2i}$$

V



$$\prod_{i=1}^{10} a_{2i-1} = \prod_{i=1}^{10} a_{2i}$$

VI

Fig. 9

have shown that (a, f, e) = (c, b, g) where we use parentheses to indicate greatest common divisors. In view of the preceding results on equal products, one wonders if the greatest divisor property also holds in more general settings.

Unfortunately, it is easy to find examples of regular hexagons with sides oriented along the main diagonals and horizontal rows of Pascal's triangle where the two alternate triads of corner coefficients have different greatest common divisors in spite of the fact that they have equal products by Theorem 3. We have such examples for hexagons with 3, 4, 5 and 6 coefficients per side and conjecture that the property only holds in general for the fundamental hexagons of Hoggatt and Hansell. Also, we observe that, for the parallelograms of Fig. 2, the products acfh and bdeg are equal but that (a, c, f, h) is not necessarily equal to (b, d, e, g). At the same time, we have been unable to find examples of hexagons of the type of Theorem 5 where the greatest common divisor of the two sets of alternate boundary coefficients are not equal. Of course, these greatest common divisors are usually equal to one, but the three regular hexagons with four elements per side whose upper left-hand coefficients are, respectively,

$$\binom{13}{6}, \quad \binom{14}{6}, \quad \text{and} \quad \binom{17}{6}$$

have pairs of greatest common divisors equal to 13, 13, and 34, respectively. We conjecture that the greatest common divisors of the two sets of alternate boundary coefficients for the hexagons of Theorem 5 are equal.

This is not true, however, of the octagons of Theorem 6, since, in particular,

$$\left(\binom{5}{1}, \quad \binom{8}{2}, \quad \binom{9}{4}, \quad \binom{6}{3}\right) = 1,$$
$$\left(\binom{6}{1}, \quad \binom{9}{3}, \quad \binom{8}{4}, \quad \binom{5}{2}\right) = 2,$$

and these are alternate boundary coefficients of such an octagon. Of course, this makes it clear that not all polygons that can be tiled with fundamental hexagons have the equal greatest common divisor property. At the same time, some figures that cannot be tiled with fundamental hexagons appear to have the equal greatest common divisor property. For example, this appears to be true of the octagon of Fig. 11 in [3] though we have no proof of this fact. This leaves the question of the characterization of figures having the equal greatest common divisor property quite open.

5. GENERALIZATIONS AND EXTENSIONS

There are an infinitude of other Pascal-like arrays in which the Hexagon Squares property holds. For example, the Fibonomial triangle and the generalized Fibonomial triangle. If, indeed we replace F_n by $f_n(x)$, the property holds and thus for each x integral yields an infinitude of such arrays. For example, if $x = 2$, we get the Pell numbers, or every k^{th} Pell Number Sequence works.

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2. V. E. Hoggatt, Jr., and W. Hansell, "The Hidden Hexagon Squares," *Fibonacci Quarterly*, Vol. 9, No. 2 (Apr. 1971), pp. 120, 133.
3. Calvin T. Long, "Arrays of Binomial Coefficients whose Products are Squares," *Fibonacci Quarterly*, Vol. 11, No. 5 (Dec. 1973), pp. 449-456.
4. Carl L. Moore, "More Hidden Hexagon Squares," *Fibonacci Quarterly*, Vol. 11, No. 5 (Dec. 1973), p. 525.

(Continued from page 46.)

that

$$F_{8\cdot 3^{n-1}} \equiv 3^n \pmod{3^{n+1}}$$

$$F_{8\cdot 3^{n-1}-1} \equiv 1 + 3^n \pmod{3^{n+1}} .$$

Therefore

$$F_{8\cdot 3^{n-1}+x} \equiv F_x + 3^n(F_x + F_{x+1}) \pmod{3^{n+1}} .$$

If x satisfies (*), then either x or $8\cdot 3^{n-1} + x$ or $16\cdot 3^{n-1} + x$ will be congruent to m modulo 3^{n+1}. Therefore (*) has solutions for arbitrarily large n.

Problem 2. The number N is said to have complete Fibonacci residues if there exists a solution to the congruence

$$F_x \equiv m \pmod{N}$$

for all integers m. A computer search shows that the only values of $N \le 500$ having complete Fibonacci residues are the divisors of

$$3^5,\ 2^2\cdot 5^3,\ 2\cdot 3\cdot 5^3,\ 5\cdot 3^4,\ \text{or}\ 7\cdot 5^3 .$$

Determine all N which have complete Fibonacci residues.

Problem 3 is submitted by the undersigned and Leonard Carlitz, Duke University, Durham, North Carolina.

Problem 3. Show that if $= e^{\pi i/n}$, then

(Continued on page 82.)