LINEARLY RECURSIVE SEQUENCES OF INTEGERS

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PART 1. INTRODUCTION

As harmless as it may appear, the Fibonacci sequence has provoked a remarkable amount of research. It seems that there is no end to the results that may be derived from the basic definition

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1,$$

which Leonardo of Pisa found lurking in the simple rabbit problem. For example, an extension of the definition yields the so-called Lucas numbers:

$$L_{n+2} = L_{n+1} + L_n \quad \text{and} \quad L_0 = 2 \quad \text{and} \quad L_1 = 1.$$

Evidently, any two integers may be used "to start" the sequence. However, it is well known that there is an extraordinary relationship between the Fibonacci and Lucas sequences. In particular:

$$F_{2n} = F_n L_n \quad \text{and} \quad F_{n+1} + F_{n-1} = L_n .$$

Precisely where does this peculiarity arise?

Then, again, many remarkable summation formulae are available. In particular, the n^{th} partial sum of the Fibonacci sequence is expressed by $F_{n+2} - 1$. The method used generally for proving such formulae is induction on the index. This involves

1. a guess provoked by the investigation of individual cases,
2. an efficient formulation of the guess, and
3. a proof by finite induction.

The drawbacks to this method are obvious. First, it depends very heavily on insight and cleverness, which qualities, while being desirable in any mathematician, do not lead to results very quickly. Second, this method is entirely inadequate for cases involving bulky formulations, and, of course, many times a result suggested by individual observation does not immediately come in convenient form. Finally, such a method is unable to relate and generalize results. Mathematics is incomplete until the specific, and perhaps surprising, facts are brought back to a generalization from which they may be <u>deduced</u>. Not only does this give a foundation to the conclusions themselves, but it enables one to draw further, unsuspected conclusions, which are beyond inductive methods. Furthermore, as a result of a generalized deduction, the formulation will be more elegant and notationally consistent.

What is required then, is a generalization of the Fibonacci sequence, discarding the incidental. At points this project will appear to be unnecessarily removed from the simplicities of the original sequence, but attempts will be made to show the connections between the more general case and the more familiar results.

DEFINITIONS

The Fibonacci sequence is based on an additive relationship between any term and the two preceding terms. In our generalization, it is necessary to exploit two aspects of this relationship: we shall make it a linear dependence, and it will involve the preceding p terms. Here, and throughout, f will note the general additive sequence:

$$f_{n+p} = a_1 f_{n+p-1} + a_2 f_{n+p-2} + \cdots + a_p f_n \qquad (n = 0, 1, 2, \cdots). \tag{1}$$

It seems essential to the spirit of these sequences that they be integral. To insure this, we must demand that the set

$$\{a_i\}_1^p$$

be integers. This set will be called the <u>spectrum</u>. But, returning to (1) and letting $n = 0$:

$$f_p = \sum_{k=1}^{p} a_k f_{p-k} , \tag{2}$$

reveals that we must specify the first p terms of the sequence in order that the others may be obtained. The set of integers

$$\{f_i\}_0^{p-1}$$

so specified will be called the <u>initial set</u>, or the initials.

It might be mentioned here that the Fibonacci sequence is obtained by letting $p = 2$ and taking the spectrum $\{1, 1\}$ and the initials $\{0, 1\}$. And the Lucas sequence has $p = 2$, spectrum $\{1, 1\}$ and initials $\{2, 1\}$.

We wish now to extend the definition (1) so that negative values for the index are allowed. Using the "back-up" approach, we obtain

$$f_{p-1} = a_1 f_{p-2} + \cdots + a_{p-1} f_0 + a_p f_{-1} ,$$

or

$$f_{-1} = \frac{1}{a_p}\left(f_{p-1} - \sum_{k=1}^{p-1} a_k f_{p-1-k}\right) .$$

Continuing, it can be seen that, for any $n = 0, 1, 2, \cdots$,

$$f_{-n} = \frac{1}{a_p}\left(f_{p-n} - \sum_{k=1}^{p-1} a_k f_{p-n-k}\right) . \tag{3}$$

Clearly, in order to maintain an integral sequence for all values of the index, positive and negative, it is necessary to take $a_p = \pm 1$. In any case, we have that $a_p^2 = 1$.

UNARY SEQUENCES ($p = 1$)

The number p of necessary initial values classifies the sequence as unary, binary, tertiary, and so on. The analysis of the unary sequences is rather trivial. The spectrum is $\{a_1\}$ and the initial set $\{f_0\}$. But since $p = 1$, we must have $a_1 = \pm 1$, so that (1) comes down to:

$$f_{n+1} = \pm f_n$$

or, immediately:

$$f_{n+1} = (\pm 1)^{n+1} f_0 .$$

In addition, it would seem altogether desirable to eliminate those sequences which can be "reduced" by dividing each term by a constant. That would leave only the primitive sequences for which if d divides f_k for each value of k, then $d = 1$. In addition, we eliminate those trivial sequences with each term zero. These conditions are met by demanding that neither the spectrum nor the initial set be all zero, and that no constant be divisible into all the spectrum or initial set. With these restrictions, we see that the unary sequences become:

$$f_k = 1, \quad \text{for all } k, \quad \text{or}$$

$$f_k = (-1)^k .$$

This simply ends all discussion of unary sequences.

ALGEBRAIC GENERATORS

One of the most common manifestations of additive recursive sequences is the power series expansion of certain functions. For example, a short calculation leads one to conclude that:

$$\frac{x}{1 - x - x^2} = \sum_{k=0}^{\infty} F_k x^k .$$

The actual derivation of this result stems directly from the definition of the Fibonacci sequence. In what follows, we will use the same derivation in a generalized form. What we want to discover is an expression for:

(4) $$\sum_{k=0}^{\infty} f_k x^k = \Phi(x),$$

where, by (1):

$$f_{n+p} = \sum_{k=1}^{p} a_k f_{n+p-k} \quad .$$

Now, we multiply (1) by x^{n+p}, and sum over the index n, so that:

$$\sum_{n=0}^{\infty} f_{n+p} x^{n+p} = \sum_{k=1}^{p} a_k x^k \sum_{n=0}^{\infty} f_{n+p-k} x^{n+p-k} \quad .$$

But, taking into account (4), we may rearrange this expression, and:

$$\Phi(x)\left(1 - \sum_{k=1}^{p} a_k x^k\right) = f_0 + \sum_{k=1}^{p-1} x^k \left(f_k - \sum_{j=1}^{k} a_j f_{k-j}\right).$$

This singularly awkward expression can be made manageable by making the somewhat arbitrary definition of $a_0 = -1$. The introduction of a_0 greatly simplifies the formulation of the required function:

(5) $$\Phi(x) = \frac{\sum_{k=0}^{p-1} x^k \sum_{j=0}^{k} a_j f_{k-j}}{\sum_{k=0}^{p} a_k x^k} \quad .$$

We need hardly say that this is the required expression, which reduces to the familiar Fibonacci power series when $p = 2$, $a_1 = a_2 = 1$, and $f_0 = 0$, $f_1 = 1$. But, further investigation of (5) leads to considerations which will be of crucial importance later. First, we remark that the denominator is a p^{th}-degree polynomial:

$$-a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p, \qquad (a_0 = -1) \quad ,$$

which will be called the spectral polynomial.

Then, with regard to the numerator, the following definition will be made:

(6) $$h_{m,k} = -\sum_{j=0}^{k} a_j f_{m+k-j} \quad \text{for} \quad 0 \le k \le p \, .$$

In other words, $h_{m,k}$ is a partial sum of terms. For example:

$$h_{m,0} = -a_0 f_m = f_m$$
$$h_{m,p} = 0 \qquad \text{cf (1).}$$
$$h_{m,p-1} = a_p f_{m-1} ,$$

and, for convenience:

(7) $$h_k = h_{0,k} = f_k - a_1 f_{k-1} - \cdots - a_k f_0 .$$

The introduction of (7) into (5) yields the remarkably concise:

(5') $$\Phi(x) = \sum_{k=0}^{\infty} f_k x^k = \frac{-\sum_{k=0}^{p-1} h_k x^k}{\sum_{k=0}^{p} a_k x^k} .$$

THE Q-SEQUENCE

In any f-sequence, it is possible to choose the initial set as any set of p consecutive terms, so that two "different" sequences may actually differ only in their indices. It seems then necessary to consider some sort of fundamental sequence. This fundamental sequence has the simplest non-trivial initial set; namely $\{0, 0, \cdots, 0, 1\}$. The Fibonacci sequence is a binary case. These sequences exist for all values of p, and they will be called <u>Q-sequences</u>. Referring to (6), we will rename the partial sums $h_{m,k}$:

(8) $$H_{m,k} = -\sum_{j=0}^{k} a_j Q_{m+k-j} \qquad \text{for} \quad 0 \le k \le p ,$$

and, from (7):

$$H_k = H_{0,k} = -\sum_{j=0}^{k} a_j Q_{k-j} ,$$

but, from the definition of Q-sequences, $Q_k = 0$ for $0 \le k \le p - 2$, and $Q_{p-1} = 1$:

$$H_k = 0, \qquad \text{for} \quad 0 \le k \le p - 2, \qquad \text{or} \quad k = p ,$$
$$H_{p-1} = 1 .$$

Using these results in (5'), we have:

$$\sum_{k=0}^{\infty} Q_k x^k = \frac{-\sum_{k=0}^{p-1} H_k x^k}{\sum_{k=0}^{p} a_k x^k} = \frac{x^{p-1}}{-\sum_{k=0}^{p} a_k x^k} \tag{9}$$

The right-hand member of (9) may be treated as a geometric series:

$$\frac{x^{p-1}}{1 - \sum_{k=1}^{p} a_k x^k} = x^{p-1}\left(\sum_{k_1=0}^{\infty} \sum_{k=0}^{p} \left(a_k x^k\right)^{k_1}\right),$$

and successive binomial expansions of the polynomial in parenthesis gives:

$$x^{p-1}\left(\sum_{k_1=0}^{\infty} \sum_{k=0}^{p} \left(a_k x^k\right)^{k_1}\right) = x^{p-1}\left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} (a_1 x)^{k_1-k_2} \sum_{k=2}^{p} \left(a_k x^k\right)^{k_2}\right)$$

and so on. After p steps, we may collect coefficients of x^{m+p-1} in (9) and equate them, obtaining:

$$Q_{m+p-1} = \sum \binom{k_1}{k_2}\binom{k_2}{k_3}\binom{k_3}{k_4}\cdots\binom{k_{p-1}}{k_p} a_1^{k_1-k_2} a_2^{k_2-k_3} \cdots a_{p-1}^{k_{p-1}-k_p} a_p^{k_p}, \tag{10}$$

where the sum is taken over all $\{k_i\}$ such that

$$\sum_{i=1}^{p} k_i = m,$$

and $m + p - 1 \geq 0$. Looking at the binary case $(p = 2)$, we discover that

$$Q_{n+1} = \sum_{i=0}^{\infty} \binom{n-i}{i} a_1^{n-2i} a_2^{i},$$

where

$$\binom{n}{k} = 0 \quad \text{for} \quad 0 < n < k, \quad \text{and} \quad n + 1 \geq 0$$

so that, for $a_1 = a_2 = 1$:

$$F_{n+1} = \sum_{i=0}^{\infty} \binom{n-i}{i}$$

which, of course, is the well-known "rising diagonal" result for Fibonacci numbers, derived from Pascal's triangle. And, for comparison, here is the ternary case:

$$Q_{n+2} = \sum_{i,j} \binom{n-i-j}{i}\binom{i}{j} a_1^{n-2i-j} a_2^{i-j} a_3^{j} \quad . \tag{12}$$

Remark. In this section, and throughout the rest, we choose to make the agreement that $\binom{n}{k} = 0$ for all $0 < n < k$. This appears a bit arbitrary, but it is used since it simplifies the summation notation. Notice that the upper index on the summation may be taken as infinity, since by our agreement the binomial coefficients vanish for large enough k. It might be pointed out that the real upper index, for example, in (11) is $[\![n/2]\!]$, that is, the greatest integer in $n/2$. The bulky notation required for (10) in particular in this form warrants using our more simplified method.

THE Q-SEQUENCE AS BASIS

The fundamental nature of the Q-sequence is clearly shown in the following argument. We return to (5'), and rearrange slightly:

$$\sum_{k=0}^{\infty} f_k x^k = \frac{-\sum_{k=0}^{p-1} h_k x^k}{\sum_{k=0}^{p} a_k x^k} = \sum_{k=0}^{p-1} h_k x^{k-p+1} \left(\frac{x^{p-1}}{\sum_{k=0}^{p} a_k x^k} \right) \quad ,$$

then, taking into account (9), we have:

$$\sum_{k=0}^{\infty} f_k x^k = \sum_{k=0}^{p-1} h_k x^{k-p+1} \left(\sum_{i=0}^{\infty} Q_i x^i \right) \quad .$$

Comparing the coefficients of x^{m-p+1} in this expression, we find that:

$$f_{m-p+1} = \sum_{k=0}^{p-1} h_k Q_{m-k} \quad . \tag{13}$$

This clearly shows the basic nature of the Q-sequence — it forms a basis set for any other sequence having the same spectrum. But a more useful formulation may be derived by considering (7), and substituting into (13):

$$f_{m-p+1} = -\sum_{k=0}^{p-1}\sum_{j=0}^{k} a_j f_{k-j} Q_{m-k} , \tag{14}$$

and then, using (8):

$$-\sum_{j=0}^{k} a_j Q_{m-p+1+k-j} = H_{m-p+1,k} ,$$

we have, from (14):

$$f_{m-p+1} = \sum_{k=0}^{p-1} H_{m-p+1,k} f_{p-k-1} , \tag{15}$$

and finally, an obvious adjustment of index leaves us with:

$$f_m = \sum_{k=0}^{p-1} H_{m,k} f_{p-1-k} \quad \text{where } H_{m,k} \text{ is given by (8) .} \tag{16}$$

Remark. For certain values of k, an alternative form of (8) is desirable:

$$H_{m,k} = Q_{m+k} - \sum_{j=1}^{k} a_j Q_{m+k-j} = \sum_{j=1}^{p} a_j Q_{m+k-j} - \sum_{j=1}^{k} a_j Q_{m+k-j}$$

or

$$H_{m,k} = \sum_{j=1}^{p-k} a_{k+j} Q_{m-j} . \tag{17}$$

THE H-SEQUENCE

What appeared in (8) to be merely notational convenience can now show more positive results. For example, a linear combination of p consecutive $H_{m,k}$ over the m index (in the spirit of (1)):

$$\sum_{j=1}^{p} a_j H_{m-j,k} = -\sum_{j=1}^{p} a_j \sum_{i=0}^{k} a_i Q_{m-j+k-i}$$

(18)
$$= -\sum_{i=0}^{k} a_i \sum_{j=1}^{p} a_j Q_{m+k-i-j}$$

$$= -\sum_{i=0}^{k} a_i Q_{m+k-i} = H_{m,k}$$

shows that $H_{m,k}$ is itself an f-sequence for any choice of k. In fact, for $k = 0$, the H-sequence reduces to the Q-sequence due to:

$$H_{m,0} = -\sum_{j=0}^{0} a_j Q_{m-j} = -a_0 Q_m = Q_m .$$

But, for any choice of k, we must have in general, that H-sequences satisfy (16), since they are f-sequences:

(19)
$$H_{m,k} = \sum_{j=0}^{p-1} H_{m,j} H_{p-1-j,k}$$

which is a remarkable formula suggestive of a whole series of important results.

PART 2. MATRIX REPRESENTATIONS

A great many of the familiar Lucas and Fibonacci identities have been shown to be related to the properties of matrices. The attempt to generalize these results for higher orders of sequence directly leads to various sorts of results depending largely on the aspect taken for generalization. But our previous work has led up to the following formulae:

(8)
$$H_{m,k} = -\sum_{j=0}^{k} a_j Q_{m+k-j} \quad \text{for} \quad 0 \le k \le p$$

and

(19)
$$H_{m,k} = \sum_{j=0}^{p-1} H_{m,j} H_{p-1-j,k}$$

as well as

(16)
$$f_m = \sum_{k=0}^{p-1} H_{m,k} f_{p-1-k} .$$

These three equations are strongly suggestive of matrix multiplication, particularly the last (16). In fact, if the following definitions are made, a singularly simple formulation may be given:

(20)
$$\underline{H}^m = \begin{pmatrix} H_{m+p-1,0} & \cdots & H_{m+p-1,p-1} \\ \vdots & & \vdots \\ H_{m,0} & \cdots & H_{m,p-1} \end{pmatrix},$$

Then:

$$\underline{H}^0 = \begin{pmatrix} H_{p-1,0} & \cdots & H_{p-1,p-1} \\ \vdots & & \vdots \\ H_{0,0} & \cdots & H_{0,p-1} \end{pmatrix} = I_p, \quad \text{the identity,}$$

and

$$\underline{H}^1 = \underline{H} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

A glance at (19) shows that the matrix $\underline{H}$ is really multiplicative; that is:

$$\underline{H}^m \underline{H}^n = \underline{H}^{m+n}$$

since (19) is merely a statement of such a multiplication, row by column. Here again, what began in (8) as mere convenience, is seen to have something of a fundamental character with regard to the recursive sequences. Now, in addition, let us define:

(21)
$$\mathring{\underline{F}}_m = \begin{pmatrix} f_{m+p-1} \\ f_{m+p-2} \\ \vdots \\ f_{m+1} \\ f_m \end{pmatrix}.$$

Finally, then, it is evident that (16) may be written in the matrix form:

(22)
$$\underline{F}_m = \underline{H}^m \mathring{\underline{F}}_0$$

A particularly useful remark may be inserted here:

(23)
$$\det \underline{H}^m = (\det \underline{H})^m = (a_p(-1)^{p+1})^m .$$

This can be seen by considering definitions (20). However, in order to maintain a sequence which has integers for all values of the index we need $a_p = \pm 1$, as was seen in (3). Hence, for any value of m, $\det \underline{H}^m = \pm 1$.

Also, the harmless observation that $\underline{H}^m \underline{H}^n = \underline{H}^{m+n}$, when compared entry for entry leads to the remarkable:

(24)
$$\sum_{k=1}^{p} H_{m+p-k,j-1} H_{n+p-i,k-1} = H_{m+n+p-i,j-1}$$

which is actually a generalization of (19).

In particular, by taking $i = p$ and $j = 1$ in (24), and recalling that $H_{m,0} = Q_m$ in (8), and then rearranging index in (24):

$$\sum_{k=0}^{p-1} H_{n,k} Q_{m+p-k-1} = Q_{m+n} \quad (25)$$

GENERAL REDUCTIONS

Rather than considering column matrices of f_k, we now extend the treatment to the square matrix, having columns given by (21):

$$\underline{F}_m = \begin{pmatrix} f_{m+2p-1} & \cdots & f_{m+p-1} \\ f_{m+2p-2} & \cdots & f_{m+p-2} \\ \vdots & & \vdots \\ f_{m+p-1} & \cdots & f_m \end{pmatrix}, \quad (26)$$

Then (22) becomes an expression involving $p \times p$ matrices:

$$\underline{H}^m \underline{F}_0 = \underline{F}_m$$

where:

$$\underline{F}_0 = \begin{pmatrix} f_{2p-1} & \cdots & f_{p-1} \\ \vdots & & \vdots \\ f_{p-1} & \cdots & f_0 \end{pmatrix},$$

Taking determinants, and simplifying, using (23):

$$\begin{aligned} \det \underline{F}_m &= \det \underline{H}^m \det \underline{F}_0 \\ &= (a_p(-1)^{p-1})^m \det \underline{F}_0 \end{aligned} \quad (27)$$

or:

$$\left|\det \underline{F}_m\right| = \left|\det \underline{F}_0\right| \quad \text{for any } m .$$

Clearly, the number $\det \underline{F}_0$ is an extraordinary constant for any sequence which depends on the initial set $\{f_i\}_0^{p-1}$, and which will be called the <u>characteristic.</u>

The characteristic of the Fibonacci sequence is

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 ,$$

while that of the Lucas sequence is

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5 ,$$

as is well known.

Again, the simple remark that:

$$\underline{F}_{m+n} = \underline{H}^{m+n}\underline{F}_0 = \underline{H}^m \underline{F}_n$$

plus, a comparison of entries, gives:

$$f_{m+n} = \sum_{k=0}^{p-1} H_{m,k} f_{n+p-1-k} \tag{28}$$

which is the <u>general reduction.</u> It is a generalization of (16).

EXAMPLES

The binary case, of course, yields the most familiar results:

$$\underline{H}^m = \begin{pmatrix} Q_{m+1} & a_2 Q_m \\ Q_m & a_2 Q_{m-1} \end{pmatrix} \tag{29}$$

so that:

$$\underline{H} = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \det \underline{H} = -a_2 .$$

From (27) in the binary case:

$$f_{m+2} f_m - f_{m+1}^2 = (-a_2)^m (f_2 f_0 - f_1^2) \tag{30}$$

so that, for the Q-sequence:

$$Q_{m+2} Q_m - Q_{m+1}^2 = (-a_2)^m (-1)$$

or

$$Q_m^2 - Q_{m+1} Q_{m-1} = (-a_2)^{m-1} \tag{31}$$

and the binary reduction becomes, referring to (28):

$$Q_m f_{n+1} + a_2 Q_{m-1} f_n = f_{m+n} . \tag{32}$$

The correspondence with the usual Fibonacci results may be worked out in detail directly from these identities.

Now, turning our attention to the ternary case $(p = 3)$, we discover several important points. First, the elegant formulations of the binary case do not hold up for $p = 3$, or for higher cases. Also, symmetry of expression begins to fade with the higher sequences.

Clearly, most of the interesting properties of the Fibonacci sequence stem from its being a binary sequence, while a few come from its being a sequence in general. We will here give the ternary results:

$$\underline{H}^m = \begin{pmatrix} Q_{m+2} & a_2Q_{m+1} + a_3Q_m & a_3Q_{m+1} \\ Q_{m+1} & a_2Q_m + a_3Q_{m-1} & a_3Q_{m-1} \\ Q_m & a_2Q_{m-1} + a_3Q_{m-2} & a_3Q_{m-2} \end{pmatrix} \tag{33}$$

and

$$\underline{H} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that: $\det \underline{H} = a_3$ hence:

$$\begin{vmatrix} f_{m+4} & f_{m+3} & f_{m+2} \\ f_{m+3} & f_{m+2} & f_{m+1} \\ f_{m+2} & f_{m+1} & f_m \end{vmatrix} = (a_3)^m \begin{vmatrix} f_4 & f_3 & f_2 \\ f_3 & f_2 & f_1 \\ f_2 & f_1 & f_0 \end{vmatrix} . \tag{34}$$

For Q-sequences in the ternary case:

$$\begin{vmatrix} Q_{m+4} & Q_{m+3} & Q_{m+2} \\ Q_{m+3} & Q_{m+2} & Q_{m+1} \\ Q_{m+2} & Q_{m+1} & Q_m \end{vmatrix} = -(a_3)^m . \tag{35}$$

And the ternary reduction:

$$Q_m f_{n+2} + (Q_{m+1} - a_1 Q_m) f_{n+1} + a_3 Q_{m-1} f_n = f_{m+n} . \tag{36}$$

NEGATIVE INDEX

Already, we have investigated the nature of the general sequence for negative values of the index. A necessary and sufficient condition that a sequence be primitive and integral is that $a_p^2 = 1$. Now, using more recent results, it is possible to look into the matter a bit more deeply, and obtain expressions relating terms of negative index with those of positive index.

Were the matrix equation $\underline{H}^m \underline{F}_0 = \underline{F}_m$ *to hold for negative value of the index:*

$$\underline{F}_{-m} = \underline{H}^{-m} \underline{F}_0 = (\underline{H}^m)^{-1} \underline{F}_0 ,$$

or, in particular:

$$\underline{H}^{-m} = (\underline{H}^m)^{-1} , \tag{37}$$

so that, after the indicated inversion, we may equate the entries in (37):

(38)
$$H_{-m+p-i,\,j-1} = \frac{1}{\det \underline{H}^m} \text{ minor } H_{m+p-j,\,i-1} .$$

Then, letting $j = 1$, and recalling (8):

$$H_{-m+p-i,\,0} = Q_{-m+p-i} = \frac{1}{\det \underline{H}^m} \text{ minor } H_{m+p-1,\,i-1}$$

and, then, letting $i = p$, we have, after reference to (23):

(39)
$$Q_{-m} = (a_p(-1)^{p+1})^{-m} \text{ minor } H_{m+p-1,\,p-1} .$$

Then, for the general case, we need only note that from (16):

(16')
$$f_{-m} = \sum_{k=0}^{p-1} H_{-m,k}\, f_{p-1-k} ,$$

where:

(8')
$$H_{-m,k} = \sum_{j=0}^{k} a_j Q_{-m+k-j} .$$

As a footnote, we add two identities coming from the equation $\underline{H}^m \underline{H}^{-m} = I_p$, where entries are compared, after completing the multiplication on the left member:

(40)
$$\sum_{k=1}^{p} H_{m+p-k,\,j-1} H_{-m+p-i,\,k-1} = H_{p-i,\,j-1} = \delta_{ij}$$

for $0 \le 1 \le p$, $0 \le j \le p$ and $0 \le k \le p$, and where δ_{ij} is Kronecker's delta.

If $i = p$ and $j = 1$ in (40):

$$\sum_{k=1}^{p} H_{m+p-k,\,0} H_{-m,\,k-1} = H_{0,0} ,$$

which may be rewritten:

(41)
$$\sum_{k=0}^{p-1} Q_{m+p-k-1} H_{-m,k} = 0 .$$

Applying (39) to the binary case yields the intriguing result:

(42) $$Q_{-m} = -(-a_2)^{-m} Q_m \quad \text{or that:} \quad |Q_{-m}| = |Q_m|, \quad \text{for} \quad p = 2,$$

while in the ternary case:

(43) $$Q_{-m} = (a_3)^{-m}(Q^2_{m+1} - Q_m Q_{m+2}), \qquad \text{for} \quad p = 3.$$

Clearly, the beauty of the expression for $p = 2$ does not carry over to the situations for greater values of p.

MATRIX SEQUENCES

An obvious, but interesting result of (26) is the matrix expression (using entrywise addition):

(44) $$\sum_{k=1}^{p} a_k \underline{F}_{m-k} = \underline{F}_m .$$

From (1), it is evident that the matrices $\{\underline{F}_k\}$ form an f-sequence with spectrum $\{a_k\}_1^p$. Furthermore, (44) may be written, using the definition of $\underline{F}_m$:

$$\sum_{k=1}^{p} a_k \underline{F}_{m-k} = \sum_{k=1}^{p} a_k \underline{H}^{m-k} \underline{F}_0 = \underline{H}^m \underline{F}_0 ;$$

however, $\underline{F}_0 \neq 0$, so that, dividing it out:

(45) $$\sum_{k=0}^{p} a_k \underline{H}^{m-k} = 0 ,$$

in which case the powers of the matrix $\underline{H}$ form an f-sequence. In fact, (45) is really a result of the Cayley-Hamilton Theorem.

ROOTS OF THE SPECTRAL POLYNOMIAL

Returning to the earlier question of explicit determination of f_m and Q_m, we recall that (10) was obtained, which expressed Q_m in terms of a sum of binomial coefficients. A different approach now will yield the so-called Binet form, which may then be compared with (10) for a series of remarkable relationships. But first we return to (5'):

$$\sum_{k=0}^{\infty} f_k x^k = \frac{\sum_{k=0}^{p-1} h_k x^k}{-\sum_{k=0}^{p} a_k x^k} \quad ;$$

recalling that the spectral polynomial appears in the denominator. Now consider that this polynomial has been factored in the complex field:

(46)
$$-\sum_{k=0}^{p} a_k x^k = \prod_{i=1}^{p} (1 - r_i x) \quad ,$$

where the roots are $\{1/r_i\}_1^p$, a set of complex numbers, none of which are zero. Now, let us make the very strong assumption that the roots are distinct, so that:

(47)
$$\sum_{k=0}^{\infty} f_k x^k = \frac{\sum_{k=0}^{p-1} h_k x^k}{\prod_{i=1}^{p} (1 - r_i x)} = \sum_{i=1}^{p} \frac{A_i}{1 - r_i x} \quad ,$$

where the right-hand member is a sum of partial fractions. What is needed is an expression for each A_i. Using a geometric series and (47):

$$\sum_{k=0}^{\infty} f_k x^k = \sum_{i=1}^{p} A_i \sum_{j=0}^{\infty} r_i^j x^j \quad ,$$

and, equating coefficients of x^k:

(48)
$$f_k = \sum_{j=1}^{p} A_j r_j^k \quad .$$

Then, from (5') and (46):

$$\sum_{k=0}^{\infty} f_k x^k \left(\prod_{i=1}^{p} (1 - r_i x) \right) = \sum_{k=0}^{p-1} h_k x^k \quad ,$$

and so, cancelling the term $(1 - r_k x)$, after introducing (47):

$$\sum_{k=1}^{p} A_k \prod_{i \neq k} (1 - r_i x) = \sum_{k=0}^{p-1} h_k x^k .$$

Now, we substitute r_n $(n = 1, 2, \cdots, p)$ for $1/x$, and recall that the $\{r_i\}$ are distinct:

$$A_n \left(\prod_{i \neq n} \left(1 - \frac{r_i}{r_n} \right) \right) = \sum_{k=0}^{p-1} h_k (1/r_n)^k$$

so that, finally:

$$A_n = \frac{\sum_{k=0}^{p-1} h_k r_n^{p-1-k}}{\prod_{i \neq n} (r_n - r_i)} \tag{49}$$

which is exactly the expression for the A_n demanded for (47), and (48). In fact, now we may introduce into (48) the value for A_j derived from (49):

$$f_m = \sum_{j=1}^{p} \sum_{k=0}^{p-1} h_k \cdot \frac{r_j^{m+p-1-k}}{\prod_{i \neq j} (r_j - r_i)}$$

or:

$$f_m = \sum_{k=0}^{p-1} h_k \left(\sum_{j=1}^{p} \frac{r_j^{m+p-1-k}}{\prod_{i \neq i} (r_j - r_i)} \right) . \tag{50}$$

However, comparing (50) with (13) and rearranging index shows that:

$$Q_m = \sum_{j=1}^{p} \frac{r_j^m}{\prod_{i \neq j} (r_j - r_i)} , \tag{51}$$

where, of course, the r_i are assumed to be distinct. This expression (51) is the <u>general Binet-form</u> for the Q-sequences.

Remark. We note that, at the opposite extreme, the assumption might have been made that $r_i = r$ for all i, so that all the spectral polynomial roots are equal:

(52) $$-\sum_{k=0}^{p} a_k x^k = (1 - rx)^p ,$$

in which case (5') becomes:

(53) $$\sum_{k=0}^{\infty} f_k x^k = \frac{\sum_{k=0}^{p-1} h_k x^k}{(1 - rx)^p}$$

and a geometric expansion, and comparison of coefficients of x^m gives:

(54) $$f_m = \sum_{j=0}^{p-1} \binom{m + p - 1 - j}{p - 1} h_j r^{m-j}$$

and, again, comparing this expression with (13) gives that:

(55) $$Q_m = \binom{m}{p - 1} r^{m-p+1} .$$

EXAMPLES

In the binary case, many of the above results produce elegant formulae. Hence, if in (51) the roots are $1/r_1 \neq 1/r_2$ and $1 - a_1x - a_2x^2 = (1 - r_1x)(1 - r_2x)$, then:

(56) $$Q_m = \frac{r_1^m}{r_1 - r_2} + \frac{r_2^m}{r_2 - r_1} = \frac{r_1^m - r_2^m}{r_1 - r_2} ,$$

where $r_1 + r_2 = a_1$ and $r_1r_2 = -a_2$.

In the case that $r_1 = r_2 = r$, we have $2r = a_1$ and $r^2 = -a_2$, so that there are two cases: 1) $r = +1$, $a_1 = 2$ and $a_2 = -1$, and 2) $r = -1$, $a_1 = -2$ and $a_2 = 1$. And, in either case:

(57) $$Q_m = \binom{m}{1} r^{m-1} = mr^{m-1} ,$$

where:

$$Q_{m+2} = 2rQ_{m+1} - Q_m .$$

Evidently, by factoring and dividing in (56), and then allowing r_1 to approach r_2:

$$Q_m = \sum_{k=0}^{m-1} r_1^{m-1-k} r_2^k = mr^{m-1} \quad \text{if} \quad r_1 = r_2 ,$$

hence, all three cases may be said to be derived from (56).

Now, foregoing the tedious calculation, we give the ternary results $(p = 3)$:

(58) $$Q_m = \frac{r_1^m}{(r_1 - r_2)(r_1 - r_3)} + \frac{r_2^m}{(r_2 - r_3)(r_2 - r_1)} + \frac{r_3^m}{(r_3 - r_1)(r_3 - r_2)} ,$$

where

$$1 - a_1x - a_2x^2 - a_3x^3 = (1 - r_1x)(1 - r_2x)(1 - r_3x) .$$

If two roots are equal, then $r_2 = r_3$ say, and:

(59) $$Q_m = \frac{r_1^m - r_2^m}{(r_1 - r_2)^2} - \frac{mr_2^{m-1}}{r_1 - r_2} ,$$

where

$$1 - a_1x - a_2x^2 - a_3x^3 = (1 - r_1x)(1 - r_2x)^2 .$$

And, if $r_1 = r_2 = r_3 = r$, then:

(60) $$Q_m = \tfrac{1}{2} m(m - 1)r^{m-2} ,$$

where

$$Q_{m+3} = 3rQ_{m+2} - 3Q_{m+1} + rQ_m \quad \text{and} \quad r = +1 \text{ or } -1 .$$

Once again, although now there are an infinite number of cases depending on the nature of the roots, it can be seen that (59) and (60) can be derived from (58) directly, using in part the identity:

$$\frac{r_1^m - r_2^m}{(r_1 - r_2)^2} = \sum_{k=2}^{m} r_1^{m-k} r_2^{k-2}(k - 1) + \frac{mr_2^{m-1}}{r_1 - r_2} .$$

In summary, then, we can, with minor adjustments in view of multiple roots of the spectral polynomial, consider that the form (51) actually is the expression for Q_m in terms of the roots of the spectral polynomial. On the other hand, (10) expresses Q_m in terms of the coefficients of the spectral polynomial. That this is a source of a multitude of fascinating problems is left to the imagination of the reader, as well as to his leisure.

PART 3. SYMMETRIC FUNCTIONS

By attacking the entire problem from another point of view, it will be possible to derive a generalization of the Lucas sequence, and thence derive a set of remarkable identities involved with this generalization similar to the usual Fibonacci-Lucas result that $F_n L_n = F_{2n}$.

Consider, first, a set of complex numbers $\{r_i\}_1^k$, and a defining relation:

$$\prod_{j=1}^{k} (x - r_j) = \sum_{k=0}^{k} (-1)^i S_i x^{k-i} \quad . \tag{61}$$

The coefficients S_i are clearly the elementary <u>symmetric functions</u> of $\{r_j\}$. In particular:

$$\begin{aligned} S_0 &= 1, \quad \text{in any case,} \\ S_1 &= r_1 + r_2 + r_3 + \cdots + r_k , \\ S_2 &= r_1 r_2 + r_1 r_3 + \cdots , \\ S_k &= r_1 r_2 r_3 \cdots r_k, \quad \text{and} \\ S_n &= 0, \quad \text{for} \quad n > k \quad . \end{aligned} \tag{62}$$

By substituting r_m into (61):

$$r_m^k = \sum_{i=1}^{k} (-1)^{i-1} S_i r_m^{k-i}$$

or, after multiplying by r_m^{n-k}:

$$r_m^n = \sum_{i=1}^{k} (-1)^{i-1} S_i r_m^{n-i} \quad . \tag{63}$$

Suppose that t_n is any linear combination of the $\{r_j^n\}$, so that from (63), it is clear that:

$$t_n = \sum_{i=1}^{k} (-1)^{i-1} S_i t_{n-i} \quad .$$

In which case, if we further define $a_i = (-1)^{i-1} S_i$, we have (letting $k = p$):

(64) $$t_n = \sum_{i=1}^{p} a_i t_{n-i} \quad .$$

Hence, from (1), a sequence of linear combinations of n^{th} powers of r_j is actually an f-sequence. Further, the Q-sequence defined by (51) is a specific case of linear combination.

It seems reasonable to investigate the properties of the simplest t-sequence, namely, the simplest linear combination of n^{th} powers of r_j, which will be called a T-sequence:

(65) $$T_n = \sum_{j=1}^{p} r_j^n$$

in which case:

$$T_0 = p$$

$$T_1 = a_1$$

$$T_2 = a_1^2 + 2a_2 \ ,$$

and, in general,

(66) $$T_k = a_1 T_{k-1} + a_2 T_{k-2} + \cdots + a_{k-1} T_1 + k a_k \qquad \text{for } k \le p \ .$$

Remark. Since $a_i = (-1)^{i-1} s_i$, then, in particular, $a_0 = -1$, as was defined earlier. In addition, it must be remarked that the a_i defined just before (64) must be integers, in keeping with the definitions made in the first part of this paper guaranteeing that the f-sequences be sequences of integers.

From (7) and (66), it can be seen that $h_{0,k} = -a_k(p - k)$ for $k \le p$ for any T-sequence. Immediately, (5') becomes

(67) $$\sum_{k=0}^{\infty} T_k x^k = \frac{\sum_{k=0}^{p-1} a_k(p - k)x^k}{\sum_{k=0}^{p} a_k x^k}$$

and, if $s(x)$ denotes the spectral polynomial

$$-\sum_{k=0}^{p} a_k x^k \ :$$

(68) $$\sum_{k=0}^{\infty} T_k x^k = p - x \cdot s'(x)/s(x) \ .$$

Already, from (9):

$$\sum_{k=0}^{\infty} Q_k x^k = x^{p-1}/s(x)$$

so that clearly, we have:

$$s(x)\frac{d}{dx}\left(x\sum_{k=0}^{\infty} Q_k x^k\right) = x^{p-1}\sum_{k=0}^{\infty} T_k x^k \tag{69}$$

in the derivation of which, a bit of the tedious rearrangement has been passed over.

In addition, noting again that $h_{0,k} = h_k = -a_k(p - k)$ and substituting into (13):

$$T_{m-p+1} = -\sum_{k=0}^{p-1} a_k(p - k)Q_{m-k}$$

$$= -p\sum_{k=0}^{p} a_k Q_{m-k} + \sum_{k=0}^{p} a_k k Q_{m-k}$$

but the first term on the right is exactly zero by (1); so:

$$T_{m-p+1} = \sum_{k=0}^{p} k a_k Q_{m-k}$$

or:

$$T_m = \sum_{k=1}^{p} k a_k Q_{m+p-1-k} \quad . \tag{70}$$

Inspecting (70) and looking at various cases leads to the remark that, in fact, (70) is exactly the generalization of $L_m = F_{m+1} + F_{m-1} = F_m + 2F_{m-1}$, which is a familiar Fibonacci-Lucas identity.

EXAMPLES

What follows now is a rather long discussion of the binary case for T-sequences. The most fascinating results occur when $p = 2$, so that a presentation of this situation is rewarding. First, in the binary case:

$$T_n = r_1^n + r_2^n, \qquad \text{where} \quad r_i^2 - a_1 r_i - a_2 = 0, \quad i = 1, 2 ;$$

or, if $r_1 = r_2 = r$:

(71) $$T_n = 2r^n \quad .$$

In either case:

$$Q_n T_n = \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)(r_1^n + r_2^n)$$
$$= \frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}$$

so that:

(72) $$Q_n T_n = Q_{2n} \quad .$$

Then, from (70):

$$T_n = a_1 Q_m + 2a_2 Q_{m-1}$$

or:

(73) $$T_n = Q_{m+1} + a_2 Q_{m-1} \quad .$$

The symmetry of (72) and (73) reveals the underlying charm of the Lucas sequence, which, of course, carries over to any binary T-sequence. Continuing, using (73) and (42):

$$\begin{aligned} T_{-m} &= Q_{-m+1} + a_2 Q_{-m-1} \\ &= -(-a_2)^{-m+1} Q_{m-1} - a_2(-a_2)^{-m-1} Q_{m+1} \\ &= (-a_2)^{-m}(a_2 Q_{m-1} + Q_{m+1}) \\ &= (-a_2)^{-m}\, T_m \end{aligned}$$

or, as in (42):

(74) $$\left| T_{-m} \right| = \left| T_m \right| \quad .$$

Applying (73), we have:

(75) $$T_{m+1} + a_2 T_{m-1} = Q_m (a_1^2 + 4a_2)$$

while the characteristic expression (27) is:

(76) $$T_{m+2} T_m - T_{m+1}^2 = (-a_2)^m (a_1^2 + 4a_2) \quad .$$

But the general reduction (28) provides the most elegant formulae, both for Q- and T-sequences:

(77) $$\begin{aligned} Q_m T_{n+1} + a_2 Q_{m-1} T_n &= T_{m+n} \\ Q_m Q_{n+1} + a_2 Q_{m-1} Q_n &= Q_{m+n} \quad , \end{aligned}$$

and, taking (74) into account:

(78) $$\begin{aligned} T_{m-n} &= (-a_2)^{-n}(Q_{m+1} T_n - Q_m T_{n+1}) \\ Q_{m-n} &= (-a_2)^{-n}(Q_{m+1} Q_n - Q_m Q_{n+1}) \end{aligned}$$

so that, adding (77) and (78):

$$(79) \qquad \begin{aligned} T_{m+n} + (-a_2)^n T_{m-n} &= T_n T_m \\ Q_{m+n} + (-a_2)^n Q_{m-n} &= T_n Q_m \end{aligned}$$

or, subtracting:

$$(80) \qquad \begin{aligned} T_{m+n} - (-a_2)^n T_{m-n} &= Q_n Q_m (a_1^2 + 4a_2) \quad \text{cf. (75)} \\ Q_{m+n} - (-a_2)^n Q_{m-n} &= Q_n T_m \end{aligned}$$

and rearranging index in (79) and (80):

$$(81) \qquad \begin{aligned} Q_{n+k} Q_{n-k} &= (T_{2n} - (-a_2)^{n-k} T_{2k}) / (a_1^2 + 4a_2) \\ Q_{n+k} T_{n-k} &= Q_{2n} + (-a_2)^{n-k} Q_{2k} \\ T_{n+k} Q_{n-k} &= Q_{2n} - (-a_2)^{n-k} Q_{2k} \\ T_{n+k} T_{n-k} &= T_{2n} + (-a_2)^{n-k} T_{2k} \end{aligned}$$

and, finally:

$$(82) \qquad Q_{2n} Q_{2k} = Q_{n+k}^2 - Q_{n-k}^2 = \frac{T_{n+k}^2 - T_{n-k}^2}{(a_1^2 + 4a_2)} .$$

Remark. The ternary and higher cases yield no such results; that is, the symmetry and conciseness do not carry over for $p > 2$. Then, it is clear, the Lucas-Fibonacci relationship is based almost entirely on the character of the two sequences as binary sequences.

PART 4. FINITE SUMS

A number of Fibonacci identities are concerned with the formulation in terms of the Fibonacci sequence of the sum of a certain series of terms of the sequence. For example, the simplest case:

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1 .$$

We now seek to generalize this result. Recalling earlier definitions and theorems:

$$(1) \qquad f_{m+p} = \sum_{k=1}^{p} a_k f_{m+p-k}$$

$$(6) \qquad h_{m,k} = -\sum_{j=0}^{k} a_j f_{m+k-j}$$

and we define: $a_0 = -1$ and $h_{0,k} = h_k$, so that:

$$\sum_{k=0}^{\infty} f_k x^k = \frac{-\sum_{k=0}^{p-1} h_k x^k}{\sum_{k=0}^{p} a_k x^k} . \tag{5'}$$

But the initial set $\{f_i\}_0^{p-1}$ may be chosen arbitrarily, so it is possible to choose for initials the set $\{f_{m+i}\}$ where $i = 0, 1, 2, \cdots, p = 1$. In that case, (5') becomes:

$$\sum_{k=0}^{\infty} f_{m+k} x^k = \frac{-\sum_{k=0}^{p-1} h_{m,k} x^k}{\sum_{k=0}^{p} a_k x^k} \tag{83}$$

and rearranging the left member:

$$\sum_{k=m}^{\infty} f_k x^{k-m} = \frac{-\sum_{k=0}^{p-1} h_{m,k} x^k}{\sum_{k=0}^{p} a_k x^k}$$

or, multiplying by x^m:

$$\sum_{k=m}^{\infty} f_k x^k = \frac{-\sum_{k=0}^{p-1} h_{m,k} x^{m+k}}{\sum_{k=0}^{p} a_k x^k} . \tag{84}$$

Then, by a simple substitution:

$$\sum_{k=n}^{\infty} f_k x^k = \frac{-\sum_{k=0}^{p-1} h_{n,k} x^{n+k}}{\sum_{k=0}^{p} a_k x^k}$$

and, subtracting these two expressions:

(85)
$$\sum_{k=m}^{n-1} f_k x^k = \frac{\sum_{k=0}^{p-1} (h_{n,k}x^n - h_{m,k}x^m)x^k}{\sum_{k=0}^{p} a_k x^k} .$$

Letting $x = 1$, and assuming that $\Sigma a_k \neq 0$:

(86)
$$\sum_{k=m}^{n-1} f_k = \frac{\sum_{k=0}^{p-1} (h_{n,k} - h_{m,k})}{\sum_{k=0}^{p} a_k} .$$

Remark. Evidently, the sum in the left member of (86) is finite, so that in the case that $\Sigma a_k = 0$, the numerator on the right must be divisible by the denominator.

In the event that $p = 2$, we have the simpler expression:

(87)
$$\sum_{k=n}^{n-1} f_k = \frac{(f_n - f_m) + a_2(f_{n-1} - f_{m-1})}{-1 + a_1 + a_2}$$

and

(88)
$$\sum_{k=1}^{n-1} Q_k = \frac{Q_n + a_2 Q_{n-1} - 1}{a_1 + a_2 - 1}$$

which reduces to the Fibonacci expression when $a_1 = a_2 = 1$.

SUMMARY

At the outset, it was proposed to find a generalization from which all the familiar results for Fibonacci-Lucas sequences might be deduced, in addition to which a consistent notation might be developed, and finally, that the sources of the peculiarity of the Fibonacci-Lucas sequences might be found. It is hoped that such proposals are worked out in the course of the paper. All that remains to be said concerns the sources of peculiarity which is the bulk of the charm surrounding the Fibonacci-Lucas sequences. Of course, some of these properties stem from the very nature of a recursive sequence of integers (such as (5) and (27)); while other properties stem from the Q-sequence in particular (for example, (10) and (51)); while others still come from those formulae which assume different forms when $a_1 = a_2 = 1$. Actually, it is quite extraordinary how many of the properties of the Fibonacci-Lucas sequences are shared by a larger class of sequences.

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