A SOLUTION OF ORTHOGONAL TRIPLES IN FOUR SUPERIMPOSED 10 x 10 x 10 LATIN CUBES

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Recently at the 78th Summer Meeting of the American Mathematical Society, Missoula, Montana (August 20-24, 1973), Professor P. Erdös and Professor E. G. Straus proposed the following classical problem to this author: Consider four digits where each digit can have a value of 0, 1, 2, ···, 9. Divide the four digits into four sets where each set contains three digits in the following way: Set A = 1st, 2nd, 3rd digits; set B = 1st, 2nd, 4th digits; set C = 1st, 3rd, 4th digits; and set D = 2nd, 3rd, 4th digits. For example: if a cell contains the four digits 3742 then 374 would belong in set A, 372 belongs in set B, 342 belongs in set C, and 742 belongs in set D.

Then, using only the digits 0, 1, 2, ···, 9, is it possible to superimpose four $10 \times 10 \times 10$ Latin Cubes such that (we consider one set at a time) set A, set B, set C, and set D will each contain in some way every one of the following 1000 three-digit numbers 000, 001, 002, ···, 999, without repetition? (It is, of course, evident there will be four digits in each and every cell of the 1000 cells.) This author has solved the above problem and we are able to construct for the first time orthogonal triples in four $10 \times 10 \times 10$ superimposed Latin Cubes.

Note. With the method of construction shown in this paper, we are also able to construct for the first time orthogonal triples in four $(4m + 2) \times (4m + 2) \times (4m + 2)$ superimposed Latin Cubes, where $3 \leq m = 3, 4, \cdots$.

In Tables 1-10, we have systematically constructed orthogonal triples in four $10 \times 10 \times 10$ superimposed Latin Cubes.

Table 1

Square Number 0

7630	6861	3405	2793	1152	8289	4014	5547	0326	9978
0796	2633	1972	4544	6321	5017	7280	9868	8409	3155
6971	5407	8639	0016	3795	4324	9548	2153	1282	7860
9408	8549	2013	1632	4284	7150	6791	0976	3865	5327
2323	0286	7540	6151	9638	1862	3975	8019	5797	4404
5287	4974	9328	7400	8869	3635	0156	1792	2543	6011
3545	1322	0866	9288	7010	2973	5637	6401	4154	8799
8159	7790	6281	3325	5977	0406	2863	4634	9018	1542
4864	3015	5157	8979	0546	9798	1402	7320	6631	2283
1012	9158	4794	5867	2403	6541	8329	3285	7970	0636

Table 2

Square Number 1

8721	5386	6649	9850	4937	3162	7473	0218	1594	2005
1854	9720	4007	7213	5596	0478	8161	2385	3642	6939
5006	0648	3722	1474	6859	7593	2215	9930	4167	8381
2645	3212	9470	4727	7163	8931	5856	1004	6389	0598
9590	1164	8211	5936	2725	4387	6009	3472	0858	7643
0168	7003	2595	8641	3382	6729	1934	4857	9210	5476
6219	4597	1384	2165	8471	9000	0728	5646	7933	3852
3932	8851	5166	6599	0008	1644	9380	7723	2475	4217
7383	6479	0938	3002	1214	2855	4647	8591	5726	9160
4477	2935	7853	0388	9640	5216	3592	6169	8001	1724

Table 3

Square Number 2

5902	3244	1718	0139	9086	4650	8895	2371	6463	7527
6133	0909	9526	8375	3464	2891	5652	7247	4710	1088
3524	2711	4900	6893	1138	8464	7377	0089	9656	5242
7717	4370	0899	9906	8655	5082	3134	6523	1248	2461
0469	6653	5372	3084	7907	9246	1528	4890	2131	8715
2651	8525	7467	5712	4240	1908	6083	9136	0379	3894
1378	9466	6243	7657	5892	0529	2901	3714	8085	4130
4080	5132	3654	1468	2521	6713	0249	8905	7897	9376
8245	1898	2081	4520	6373	7137	9716	5462	3904	0659
9896	7087	8135	2241	0719	3374	4460	1658	5522	6903

Table 4

Square Number 3

9873	4509	7232	6317	0490	2026	5948	1755	3181	8664
3311	6877	0660	5758	4189	1945	9023	8504	2236	7492
4669	1235	2876	3941	7312	5188	8754	6497	0020	9503
8234	2756	6947	0870	5028	9493	4319	3661	7502	1185
6187	3021	9753	4499	8874	0500	7662	2946	1315	5238
1025	5668	8184	9233	2506	7872	3491	0310	6757	4949
7752	0180	3501	8024	9943	6667	1875	4239	5498	2316
2496	9313	4029	7182	1665	3231	6507	5878	8944	0750
5508	7942	1495	2666	3751	8314	0230	9183	4879	6027
0940	8494	5318	1505	6237	4759	2186	7022	9663	3871

Table 5

Square Number 4

0064	2417	4551	8278	6345	5993	3109	7626	9832	1780
9272	8068	6785	3629	2837	7106	0994	1410	5553	4341
2787	7556	5063	9102	4271	3839	1620	8348	6995	0414
1550	5623	8108	6065	3999	0344	2277	9782	4411	7836
8838	9992	0624	2347	1060	6415	4781	5103	7276	3559
7996	3789	1830	0554	5413	4061	9342	6275	8628	2107
4621	6835	9412	1990	0104	8788	7066	2557	3349	5273
5343	0274	2997	4831	7786	9552	8418	3069	1100	6625
3419	4101	7346	5783	9622	1270	6555	0834	2067	8998
6105	1340	3279	7416	8558	2627	5833	4991	0784	9062

Table 6

Square Number 5

4255	1693	5880	3426	7574	6308	2762	9039	8917	0141
8427	3256	7144	2032	1913	9769	4305	0691	6888	5570
1143	9889	6258	8767	5420	2912	0031	3576	7304	4695
0881	6038	3766	7254	2302	4575	1423	8147	5690	9919
3916	8307	4035	1573	0251	7694	5140	6768	9429	2882
9309	2142	0911	4885	6698	5250	8577	7424	3036	1763
5030	7914	8697	0301	4765	3146	9259	1883	2572	6428
6578	4425	1303	5910	9149	8887	3696	2252	7761	7034
2692	5760	9579	6148	8037	0421	7884	4915	1253	3306
7764	0571	2422	9699	3886	1033	6918	5300	4145	8257

Table 7

Square Number 6

6446	0122	2364	7985	8613	1777	9531	3800	4058	5299
4988	7445	8293	9801	0052	3530	6776	5129	1367	2614
0292	3360	1447	4538	2984	9051	5809	7615	8773	6126
5369	1807	7535	8443	9771	6616	0982	4298	2124	3050
7055	4778	6806	0612	5449	8123	2294	1537	3980	9361
3770	9291	5059	6366	1127	2444	4618	8983	7805	0532
2804	8053	4128	5779	6536	7295	3440	0362	9611	1987
1617	6986	0772	2054	3290	4368	7125	9441	5539	8803
9121	2534	3610	1297	4808	5989	8363	6056	0442	7775
8533	5619	9981	3120	7365	0802	1057	2774	6296	4448

Table 8

Square Number 7

3397	7738	0173	1041	2829	9514	6680	8962	5205	4456
5045	1391	2459	6960	7208	8682	3517	4736	9174	0823
7458	8172	9394	5685	0043	6200	4966	1821	2519	3737
4176	9964	1681	2399	6510	3827	7048	5455	0733	8202
1201	5515	3967	7828	4396	2739	0453	9684	8042	6170
8512	6450	4206	3177	9734	0393	5825	2049	1961	7688
0963	2209	5735	4516	3687	1451	8392	7178	6820	9044
9824	3047	7518	0203	8452	5175	1731	6390	4686	2969
6730	0683	8822	9454	5965	4046	2179	3207	7398	1511
2689	4826	6040	8732	1171	7968	9204	0513	3457	5395

Table 9

Square Number 8

2118	8950	9097	4562	5701	0845	1226	6484	7679	3333
7569	4112	5331	1486	8670	6224	2848	3953	0095	9707
8330	6094	0115	7229	9567	1676	3483	4702	5841	2958
3093	0485	4222	5111	1846	2708	8560	7339	9957	6674
4672	7849	2488	8700	3113	5951	9337	0225	6564	1096
6844	1336	3673	2098	0955	9117	7709	5561	4482	8220
9487	5671	7959	3843	2228	4332	6114	8090	1706	0565
0705	2568	8840	9677	6334	7099	4952	1116	3223	5481
1956	9227	6704	0335	7489	3563	5091	2678	8110	4842
5221	3703	1566	6954	4092	8480	0675	9847	2338	7119

Table 10

Square Number 9

1589	9075	8926	5604	3268	7431	0357	4193	2740	6812
2600	5584	3818	0197	9745	4353	1439	6072	7921	8266
9815	4923	7581	2350	8606	0747	6192	5264	3438	1079
6922	7191	5354	3588	0437	1269	9605	2810	8076	4743
5744	2430	1199	9265	6582	3078	8816	7351	4603	0927
4433	0817	6742	1929	7071	8586	2260	3608	5194	9355
8196	3748	2070	6432	1359	2814	4583	9925	0267	7601
7261	1609	9435	8746	4813	2920	5074	0587	6352	3198
0077	8356	4263	7811	2190	6602	3928	1749	9585	5434
3358	6262	0607	4073	5924	9195	7741	8436	1819	2580

Proof that Construction is Correct. Before going on with the proof, we will set down a few definitions to facilitate our explanation of the proof. It will be noted that the squares in Tables 1-10 are labeled Square 0 through 9. Then suppose we wish to find a certain number of a certain cell — we shall write S (row number, column number, square number) = number in cell. To find a row on a certain square, we write S (row number, *, square number), and S (*, c, s) = column number on a certain square.

The ten columns in each square are considered to be numbered 0, 1, ···, 9 from left to right; the ten rows on each square are considered to be numbered 0, 1, ···, 9 from top to bottom. For example: The number 7630 on Square Number 0 = S(0,0,0); or the row on which 7630 is found may be written as S(0, *, 0); and the column we find 7630 in is S(*, 0, 0). Finally if we refer to a specific square, say square 0, we write S(*,*, 0); if we refer to each and every one of the ten squares we write S(*, *, A); to refer to each and every top row (say) in each and every one of the two squares we write S(0, *, A).

(1) We now consider the 2nd and 3rd digits in each cell of the S(0, *, A), and keeping the cells in the same positions, we place S(0, *, 0), on top of S(0, *, 1), ···, on top of S(0, *, 9) it is easily verified that we have constructed the following 10 × 10 square which was formed by superimposing two Latin Squares in such a way that the 100 two-digit numbers are mutually orthogonal.

(1a)

63	86	40	···	97
72	38	64	···	00
···	···	···	···	···
58	07	92	···	81

(1b) It should also be noticed that the 2nd and 3rd digits in each cell of the S(0, *, A) is repeated ten times in its own respective square. For example: The ten cells of 2nd and 3rd digits in S(0, *, 0) are 63 86 40 ··· 97, and it is seen that in the Square 0, the number 63 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column, the number 86 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column, ···, the number 97 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column.

(1c) Now it is easily verified; each and every one of the ten Squares is constructed in the exact way we constructed the Square Number 0 in (1b).

(2) We now look at the first digit in each cell, where it is easily verified that the first digit in each cell of the S(0, *, A) is repeated ten times in a different row and different column on its own respective square.

(2a) For example: the first digit 0 on Square 0 will be found in ten different cells where each cell is in a different row and different column, and this exact arrangement of the first digit 0 is constructed into each and every square 0 through and including Square 9. It is also easily verified that each first digit 0 is on a different file.

(2b) Now, each and every first digit (0, 1, ⋯, 9) in every cell is arranged in the exact way we placed the 0's in our example (2a).

Therefore, there are no two identical first digits in the same row, the same column, or the same file throughout the cube.

(Let the 100 numbers 000, 001, 002, ⋯, 099 = a_0;

the 100 numbers 100, 101, 102, ⋯, 199 = a_1;

. . . .

the 100 numbers 900, 901, 902, ⋯, 999 = a_9.)

Now, combining (1, a, b, c) with (2, a, b) leads to

(3) The first three digits in each cell in the cube that belongs to ---a_k will have each of its three-digit numbers in a different column, different row, and in a different file, where we replace the subscript k (in a_k) one at a time with the number 0, then 1, ⋯, then 9.

(3a) In (3), we have then satisfied the requirement that set A (set A = the 1st, 2nd, and 3rd digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ⋯, 999, without repetition.

(3b) We now combine in each cell throughout the cube—the second and third digits with the fourth digit — and in the exact way we found (3a) — we find that we have satisfied the requirement that set D (set D = the 2nd, 3rd, and 4th digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ⋯, 999, without repetition.

(4) Now, it will be noticed that every identical first digit is paired with an identical fourth digit — we inspect one square at a time. For example: In Square 0, every one of the ten cells that have a first digit 0 also have as a fourth digit the number 6; every one of the ten cells that have a first digit 1 also have as a fourth digit the number 2; ⋯; every one of the ten cells that have a first digit 9 also have as a fourth digit the number 8. It should also be noticed that the ten first digits (say 1st digit = A) paired with ten fourth digits (say) B to get the numbers A--B in ten cells on a particular square — shall never again have this particular first and fourth digit combination repeated (that is, the combination A--B) on any one of the nine remaining squares. For example: on Square 0 the first digit 7 is paired with the fourth digit 0, on Square 1 the first digit 7 is paired with the fourth digit 3, ⋯, on Square 9, the first digit 7 is paired with the fourth digit 1. This arrangement for first and fourth digits is ridgidly enforced throughout the construction.

(5) Now, the first and second digits in each square (we consider one square at a time) are mutually (pairwise) orthogonal. For example: The first and second digits in Square 0 are mutually orthogonal and are constructed by superimposing two 10×10 Latin Squares.

(5a) The exact orthogonal properties of digits 1 and 2 in each of the ten squares (we consider one square at a time) that we find to hold true in (5) also are easily verified to hold true for the first and third digits. That is, the first and third digits in each and every one of the ten squares (we consider one square at a time) are mutually (pairwise) orthogonal.

(6) Now, we combine (4) and (5), which leads us to the fact that set B (set B = 1st, 2nd, and 4th digits in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ···, 999, without repetition.

(6a) Finally, we combine (4) and (5a), which leads us to the fact that set C (set C = 1st, 3rd, and 4th digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ···, 999, without repetition.

Remark. We used The Arkin-Hoggatt method [1] to get the 100 mutually **orthogonal** numbers in (1).

Note. For singly-even cubes greater than $10 \times 10 \times 10$ we can combine the above methods with Bose, Shrikande and Parker's work on mutually (pairwise) orthogonal numbers [2] and after the proper extensions of their magnificent theorems — it is easily shown that we can obtain a solution of orthogonal triples in four $(4m + 2) \times (4m + 2) \times (4m + 2)$ superimposed Latin Cubes (where $2 < m = 3, 4, \cdots$).

In conclusion, we discuss (our discussion relies entirely on the construction in this paper) orthogonal triples in Five $10 \times 10 \times 10$ superimposed Latin Cubes.

(7) In our discussion, the ten numbers 7630, 7860, 7400, 7790, 7150, 7280, 7010, 7540, 7320, 7970, that are found in Square Number 0 will be used as an illustrative example.

It is evident that in each of the ten numbers above, the first and fourth digits form the two-digit number 70, and also the second and third digits in the above ten numbers are mutually (pairwise) orthogonal.

(7a) Now, let us add a fifth digit to each of the ten four-digit numbers written above. It is evident that it would be impossible to form orthogonal triples if any two of the ten fifth digits we placed are identical. For example: Say we placed a 0 after (in the fifth position) two of the ten numbers in (7) — say the two numbers are 7630 and 7280. We then have 76300 and 72900 and it is evident that the 700 in 76300 and the 700 in 72800 **are** not in a set of orthogonal triples. Therefore, every one of the ten fifth digits we add to the ten numbers in (7) above must be different and thus the fifth digit in (7) must include each number in 0, 1, ···, 9. However, since the second and third digits in each of the ten numbers in (7) are mutually (pairwise) orthogonal, it follows that the second, third, and fifth digits in the above ten numbers in (7) are mutually (pairwise) orthogonal.

Then, using the exact method of our example in (7a) we extend our reasoning (step-by-step) to include the entire Square 0, and then Square 1, ···, and Square 9. In this way, we are easily led to the following.

(7b) IN ORDER TO FIND A SOLUTION OF ORTHOGONAL TRIPLES IN FIVE $10 \times 10 \times 10$ SUPERIMPOSED LATIN CUBES, WE MUST FIRST BE ABLE TO CONSTRUCT A SYSTEM OF THREE MUTUALLY ORTHOGONAL NUMBERS (three pairwise orthogonal) IN A SQUARE MADE OF THREE SUPERIMPOSED $10 \times 10 \times 10$ LATIN SQUARES.

(8) It is easily verified that by combining the NOTE above with (7b), we extend (7b) to read: IN ORDER TO FIND A SOLUTION OF ORTHOGONAL TRIPLES IN FIVE $(4m + 2) \times (4m + 2) \times (4m + 2)$ SUPERIMPOSED CUBES, WE MUST FIRST BE ABLE TO CONSTRUCT A SYSTEM OF THREE MUTUALLY ORTHOGONAL NUMBERS (three pairwise orthogonal) IN A

SQUARE MADE OF THREE SUPERIMPOSED $(4m+2) \times (4m+2) \times (4m+2)$ LATIN SQUARES, where $2 < m = 3, 4, \cdots$.

Remark. It should be noted that the methods of construction of the cube in the above paper are the same methods that were used to construct the cubes in the following two papers (we mention the following two papers, since each paper stated that a method of construction was forthcoming). See [3] and [4].

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