AN EXPANSION OF e^x OFF ROOTS OF ONE

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The proposition below is proved in [1].

Let Δ be the operator on arithmetical functions such that

$$\Delta F(n) = \sum_{d|n} F(d) . \tag{1}$$

Let

$$\sum_{n=0}^{\infty} x^n \Delta f(n)$$

converge. Let

$$\prod_{n=1}^{\infty} (1 - x^n)^{\frac{f(n)}{n}} = \sum_{n=0}^{\infty} R_f(n)\, x^n . \tag{2}$$

Then for all n:

$$0 = n R_f(n) + \sum_{a=1}^{n} \Delta f(a)\, R_f(n - a) \tag{3}$$

when x is not a root of one.

Now, let $f = \mu$ (the Mobius function) and let

$$\eta \begin{array}{l} = 1 \text{ on } 1, \\ = 0, \text{ elsewhere} . \end{array}$$

It is well known that $\Delta\mu = \eta$. Now, $\Sigma x^n \eta(n)$ converges. It follows immediately (by induction) from (3) that $R_\mu(n) = (-1)^n/n!$ and hence that

$$\prod_{n=1}^{\infty} (1 - x^n)^{\mu(n)/n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$$

(when x is not a root of 1); thus

$$e^x = \prod_{n=1}^{\infty} (1 - x^n)^{\frac{-\mu(n)}{n}}$$

off roots of 1.

REFERENCE

1. Barry Brent, "Functional Equations with Prime Roots from Arithmetical Expressions for $\mathcal{G}_\alpha$," Fibonacci Quarterly, Vol. 12, No. 2 (April 1974), pp. 199-207.