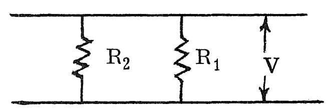
A PRIMER FOR THE FIBONACCI NUMBERS: PART XIV

V. E. HOGGATT, JR., and MARJORIE BICKNELL
San Jose State University, San Jose, California 95192

THE MORGAN-VOYCE POLYNOMIALS

1. INTRODUCTION

Polynomial sequences often occur in solving physical problems. The Morgan-Voyce polynomial results when one considers a ladder network of resistances [1], [2], [3]. Let R be the resistance of two resistors R_1 and R_2 in parallel. The voltage drop V across a resistance R due to flow of current I is, of course, $V = IR$.



Now

$$V = I_1R_1 = I_2R_2 = (I_1 + I_2)R$$

Thus

$$\frac{I_1}{V} = R_1, \qquad \frac{I_2}{V} = R_2 ,$$

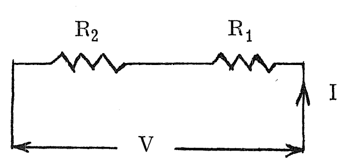
so that

$$\frac{1}{R} = \frac{I_1}{V} + \frac{I_2}{V} = \frac{1}{R_1} + \frac{1}{R_2} .$$

Thus the formula for resistors in parallel is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} .$$

For resistors in series



$$V = I(R_1 + R_2) = IR$$

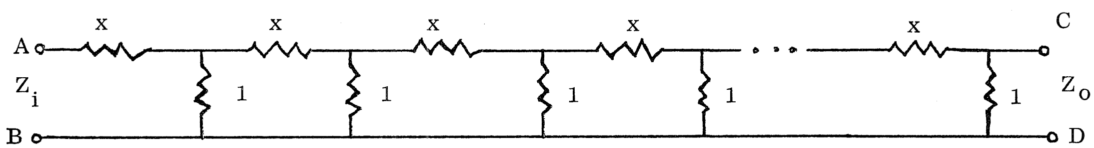
so that the formula relating the resistances is

$$R = R_1 + R_2 .$$

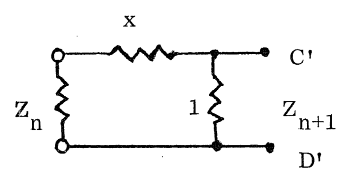
This is all we need to solve the ladder network problem.

2. LADDER NETWORKS

Consider the following:



Assume that the terminals A and B are open. We desire the resistance as measured across terminals C and D. For n ladder sections, let us assume that the resistance is Z_n, and consider the output Z_o .



Since x and Z_n are in series,

$$R = x + Z_n .$$

Now R and 1 are in parallel, so that

$$\frac{1}{Z_{n+1}} = \frac{1}{x + Z_n} + 1 = \frac{x + Z_n + 1}{x + Z_n}$$

$$Z_{n+1} = \frac{x + Z_n}{x + Z_n + 1} \quad .$$

To see what this means, let $Z_n = b_n(x)/B_n(x)$, where $b_n(x)$ and $B_n(x)$ are polynomials.

$$\frac{b_{n+1}(x)}{B_{n+1}(x)} = \frac{x + b_n(x)/B_n(x)}{x + 1 + b_n(x)/B_n(x)} = \frac{xB_n(x) + b_n(x)}{(x + 1)B_n(x) + b_n(x)}$$

so that

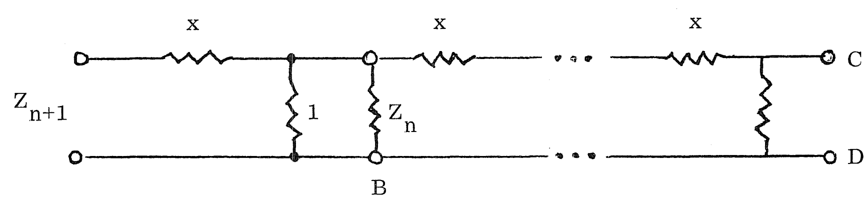
$$(2.1)\qquad \begin{cases} b_{n+1}(x) = xB_n(x) + b_n(x) \\ B_{n+1}(x) = (x+1)B_n(x) + b_n(x) \end{cases}$$

which is a mixed recurrence relation for the two polynomial sequences. Clearly, $Z_0 = 1$, so we set $b_0(x) = 1$ and $B_0(x) = 1$. This completely specifies the two sequences which we call the Morgan-Voyce polynomials.

Without too much trouble, one can derive that both sequences $\{b_n(x)\}$ and $\{B_n(x)\}$ satisfy

$$(2.2)\qquad U_{n+2}(x) = (x+2)U_{n+1}(x) - U_n(x)\,.$$

This takes care of the resistance as seen from the output end of the ladder network. We now go to the input end, and consider input Z_i.



$$\frac{1}{R} = \frac{1}{Z_n} + \frac{1}{1}, \qquad \text{or,} \qquad R = \frac{Z_n}{Z_n + 1}$$

$$Z_{n+1} = x + \frac{Z_n}{Z_n + 1} = \frac{xZ_n + x + Z_n}{Z_n + 1}\,.$$

Again let $Z_n = P_n(x)/Q_n(x)$. Then,

$$\frac{P_{n+1}(x)}{Q_{n+1}(x)} = \frac{x(P_n(x) + Q_n(x)) + P_n(x)}{P_n(x) + Q_n(x)}\,.$$

That is,

$$P_{n+1}(x) = (x+1)P_n(x) + xQ_n(x)\,,$$

$$Q_{n+1}(x) = P_n(x) + Q_n(x)\,.$$

Simplifying,

$$P_n(x) = Q_{n+1}(x) - Q_n(x)$$

$$Q_{n+2}(x) - Q_{n+1}(x) = (x+1)(Q_{n+1}(x) - Q_n(x)) + xQ_n(x)$$

or

$$Q_{n+2}(x) = (x+2)Q_{n+1}(x) - Q_n(x)\,.$$

From the case $n = 1$, we see that $P_1(x) = x + 1$, $Q_1(x) = 1$, $Q_2(x) = x + 2$, so that $Q_n(x) \equiv B_n(x)$ from the output considerations earlier, and

$$P_n(x) = Q_{n+1}(x) - Q_n(x) = B_{n+1}(x) - B_n(x) \;.$$

But, recalling the defining equation (2.1) for the Morgan-Voyce polynomials, a simple subtraction gives us $b_{n+1}(x) = B_{n+1}(x) - B_n(x)$. Thus, $P_n(x) \equiv b_{n+1}(x)$ so that

$$Z_n = \frac{b_{n+1}(x)}{B_n(x)} \;,$$

where $b_n(x)$ and $B_n(x)$ are the Morgan-Voyce polynomials. This is the resistance as seen looking into the ladder network from the input end.

There are now several theorems we can prove.

3. THEORETICAL CONSIDERATIONS

Using the recursion (2.2) for $b_n(x)$ and $B_n(x)$, it is a simple matter to compute the first few Morgan-Voyce polynomials.

n	$b_n(x)$	$B_n(x)$
0	1	1
1	$x + 1$	$x + 2$
2	$x^2 + 3x + 1$	$x^2 + 4x + 3$
3	$x^3 + 5x^2 + 6x + 1$	$x^3 + 6x + 10x + 4$
4	$x^4 + 7x^3 + 15x^2 + 10x + 1$	$x^4 + 8x^3 + 21x^2 + 20x + 5$
5	$x^5 + 9x^4 + 28x^3 + 35x^2 + 15x + 1$	$x^5 + 10x^4 + 36x^3 + 56x^2 + 35x + 6$

$$b_{n+2}(x) = (x + 2)b_{n+1}(x) - b_n(x)$$
$$B_{n+2}(x) = (x + 2)B_{n+1}(x) - B_n(x) \;.$$

Comparing these polynomials to the Fibonacci polynomials $f_n(x)$, $f_0(x) = 0$, $f_1(x) = 1$, $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$, leads to some fascinating results.

FIBONACCI POLYNOMIALS

n	$f_n(x)$										
			1								
1	1	f_1	1	1							
2	x	f_2	1	2	1						
3	$x^2 + 1$	f_3	1	3	3	1					
4	$x^3 + 2x$	f_4	1	4	6	4	1				
5	$x^4 + 3x^2 + 1$	f_5	1	5	10	10	5	1			
6	$x^5 + 4x^3 + 3x$	f_6	1	6	15	20	15	6	1		
7	$x^6 + 5x^4 + 6x^2 + 1$	f_7	1	7	21	35	35	21	7	1	
8	$x^7 + 6x^5 + 10x^3 + 4x$	f_8									

Theorem 3.1. See [3], [5]. The polynomial sequences $\{b_n(x)\}$, $\{B_n(x)\}$, and $\{f_n(x)\}$ are related by

$$f_{2n}(x) = xB_{n-1}(x^2)$$
$$f_{2n+1}(x) = b_n(x^2) .$$

Proof 1. By Generating Functions.

It is not difficult to show that

$$\frac{1-\lambda}{1-(x+2)\lambda+\lambda^2} = \sum_{n=0}^{\infty} b_n(x)\lambda^n$$

$$\frac{\lambda}{1-(x+2)\lambda+\lambda^2} = \sum_{n=0}^{\infty} B_{n-1}(x)\lambda^n .$$

Hence,

$$\frac{\lambda(1-\lambda^2)}{1-(x^2+2)\lambda^2+\lambda^4} = \sum_{n=0}^{\infty} b_n(x^2)\lambda^{2n+1}$$

$$\frac{\lambda^2 x}{1-(x^2+2)\lambda^2+\lambda^4} = \sum_{n=0}^{\infty} xB_{n-1}(x^2)\lambda^{2n} .$$

Adding these gives

$$\frac{\lambda(1+\lambda x-\lambda^2)}{1-2\lambda^2+\lambda^4-x^2\lambda^2} = \frac{\lambda}{1-x\lambda-\lambda^2} = \sum_{n=0}^{\infty} f_n(x)\lambda^n ,$$

where we recognized the generating function for the Fibonacci polynomials $\{f_n(x)\}$.

Proof 2. By Binét Forms.

Since the Fibonacci polynomials have the auxiliary equation

$$Y^2 = xY + 1 ,$$

which arises from the recurrence relation and which has roots

$$\alpha = \frac{x+\sqrt{x^2+4}}{2}, \qquad \beta = \frac{x-\sqrt{x^2+4}}{2} ,$$

it can be shown by mathematical induction that the Fibonacci polynomials have the Binét form

$$f_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta) .$$

Similarly, from the recurrence relation for the Morgan-Voyce polynomials, we have the auxiliary equation

$$Y^2 = (x + 2)Y - 1$$

with roots

$$r = \frac{x + 2 + \sqrt{x^2 + 4x}}{2}, \qquad s = \frac{x + 2 - \sqrt{x^2 + 4x}}{2},$$

leading to, via mathematical induction,

$$B_{n-1}(x) = (r^n - s^n)/(r - s) .$$

Then,

$$f_{2n}(x) = (\alpha^{2n} - \beta^{2n})/(\alpha - \beta) = \left[(\alpha^2)^n - (\beta^2)^n\right]/(\alpha - \beta)$$

$$= \left[\left(\frac{x^2 + 2 + x\sqrt{x^2 + 4}}{2}\right)^n - \left(\frac{x^2 + 2 - x\sqrt{x^2 + 4}}{2}\right)^n\right] \Big/ \sqrt{x^2 + 4} .$$

On the other hand,

$$B_{n-1}(x^2) = \left[\left(\frac{x^2 + 2 + \sqrt{x^4 + 4x^2}}{2}\right)^n - \left(\frac{x^2 + 2 - \sqrt{x^4 + 4x^2}}{2}\right)^n\right] \Big/ \sqrt{x^4 + 4x^2}$$

Notice that, since $\sqrt{x^4 + 4x^2} = |x|\sqrt{x^2 + 4}$,

$$xB_{n-1}(x^2) = f_{2n}(x) .$$

Since $b_{n+1}(x) = B_{n+1}(x) - B_n(x)$,

$$xb_{n+1}(x^2) = xB_{n+1}(x^2) - xB_n(x^2)$$

$$= f_{2n+4}(x) - f_{2n+2}(x) = xf_{2n+3}(x) ,$$

leading to

$$b_{n+1}(x^2) = f_{2n+3}(x) \quad \text{or} \quad b_n(x^2) = f_{2n+1}(x) .$$

Proof 3. By the Recurrence Relations.

Observe that

$$b_0(x) = 1, \quad b_1(x) = x + 1, \quad b_{n+2}(x) = (x + 2)b_{n+1}(x) - b_n(x) ;$$

$$f_1(x) = 1, \quad f_3(x) = x^2 + 1, \quad f_{2n+5}(x) = (x^2 + 2)f_{2n+3}(x) - f_{2n+1}(x) .$$

Thus,

$$b_0(x^2) = 1, \quad b_1(x^2) = x^2 + 1, \quad b_{n+2}(x^2) = (x^2 + 2)b_{n+1}(x^2) - b_n(x^2) .$$

Now, the sequences $\{b_m(x^2)\}$ and $\{f_{2m+1}(x)\}$ have both the same starting pair and the same recurrence relation so that they are the same sequence. Similarly,

$$B_0(x) = 1, \quad B_1(x) = x + 2 , \quad B_{n+2}(x) = (x + 2)B_{n+1}(x) - B_n(x) ;$$

$$f_2(x) = x, \quad f_4(x) = x^3 + 2x, \quad f_{2n+6}(x) = (x^2 + 2)f_{2n+4}(x) - f_{2n}(x) .$$

Next,

$$xB_0(x^2) = x, \quad xB_1(x^2) = x^3 + 2x, \quad xB_{n+2}(x^2) = (x^2 + 2)xB_{n+1}(x^2) - xB_n(x^2),$$

so that the sequences $\{xB_{n-1}(x^2)\}$ and $\{f_{2n}(x)\}$ are the same.

Several results follow immediately by applying known properties of the Fibonacci polynomials. (See [3], [6], [7].)

Corollary 3.1.1.

$$b_n(1) = F_{2n+1} \quad \text{and} \quad B_{n-1}(1) = F_{2n}$$

for the Fibonacci numbers F_n.

Corollary 3.1.2. The coefficients of $b_n(x)$ and $B_n(x)$ lie on adjacent rising diagonals of Pascal's triangle.

Corollary 3.1.3. The polynomials $\{b_n(x)\}$ are irreducible if and only if $2n + 1$ is a prime.

4. FURTHER PROPERTIES OF MORGAN-VOYCE POLYNOMIALS

Let

$$Q = \begin{pmatrix} x + 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$Q^2 = \begin{pmatrix} x + 2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x + 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x^2 + 4x + 3 & -(x + 2) \\ x + 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} B_3(x) & -B_2(x) \\ B_2(x) & -B_1(x) \end{pmatrix}.$$

It can be proved by induction [10] that

$$Q^n = \begin{pmatrix} B_{n+1}(x) & -B_n(x) \\ B_n(x) & -B_{n-1}(x) \end{pmatrix}.$$

Then, since $\det Q^n = (\det Q)^n$,

$$B_{n+1}(x)B_{n-1}(x) - B_n^2(x) = -1.$$

Thus, one can write much by virtue of having $B_n(x)$ trapped in a matrix.

Let

$$R = \begin{pmatrix} x + 2 & -2 \\ 2 & -(x + 2) \end{pmatrix}, \quad RQ^n = \begin{pmatrix} C_{n+1}(x) & -C_n(x) \\ C_n(x) & -C_{n-1}(x) \end{pmatrix},$$

so that

$$C_{n+1}(x)C_{n-1}(x) - C_n^2(x) = -(x^2 + 4x + 4) + 4 = -(x^2 + 4x).$$

Then, $C_n(x)$ corresponds to the Lucas sequence.

Let $\{L_n(x)\}$ be the Lucas polynomial sequence, $L_0(x) = 2$, $L_1(x) = x$, $L_2(x) = x^2 + 2$, $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$. Actually,

$$L_n(x) = f_{n+1}(x) + f_{n-1}(x) ,$$

and for $x = 1$, $L_n(1) = L_n$, the n^{th} member of the Lucas sequence 1, 3, 4, 7, 11, 18, 29, ⋯.

Now, $C_{-1}(x) = 2$, $C_0(x) = 2$, $C_1(x) = x + 2$. Thus, since

$$L_{2n+4}(x) = (x^2 + 2)L_{2n+2}(x) - L_{2n}(x) ,$$

we have $L_{2n}(x) = C_{n-1}(x^2)$, and $C_{n-1}(1) = L_{2n}$, a Lucas number with even subscript. Also, since

$$L_{2n}(x) = f_{2n+1}(x) + f_{2n-1}(x), \quad \text{and} \quad f_{2n+1}(x) = b_n(x^2),$$

the relationship $L_{2n}(x) = C_{n-1}(x^2)$ implies that

$$C_n(x) = b_n(x) + b_{n+1}(x) .$$

Also,

$$xB_n(x) = b_{n+1}(x) - b_n(x) ,$$

so that

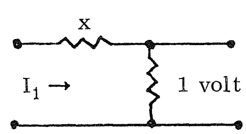
$$b_{n+1}(x) = [C_n(x) + xB_n(x)]/2 .$$

Finally, applying the divisibility properties of Lucas polynomials [6], [8], [9], we have the

Theorem. $C_{2^n}(x)$ is irreducible.

5. ATTENUATION RESULTS

The attenuation is the ratio of input voltage V_I to output voltage V_O. Since the system is linear, we can assume that the output voltage is 1V. Let us start with no resistive network. There is no current ($I_O = 0$) and between the terminals is 1 volt ($V_O = 1$).



$I_1 = 1$ amp $\qquad V_1 = x + 1$ volts

So we see that

$$I_0 = 0 = B_{-1}(x), \qquad V_0 = 1 = b_{-1}(x),$$
$$I_1 = 1 = B_0(x), \qquad V_1 = 1 = B_0(x).$$

We shall see that

$$I_n = B_{n-1}(x) \quad \text{and} \quad V_n = b_{n-1}(x).$$

First, we note that from $b_{n+1}(x) = xB_n(x) + b_n(x)$ and from

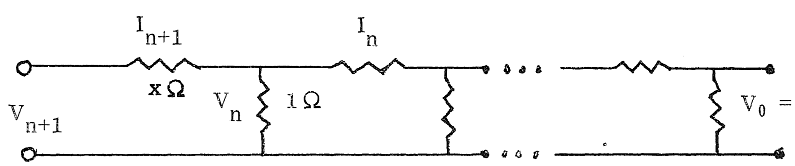
$$B_{n+1}(x) = (x+1)B_n(x) + b_n(x) = B_n(x) + xB_n(x) + b_n(x),$$

we have the lemma,

Lemma 1.

$$B_{n+1}(x) = B_n(x) + b_{n+1}(x).$$

In the ladder network, the voltage across the n^{th} unit resistance is V_n; hence, the current is also V_n.



Now, the voltage currents obey

$$V_{n+1} = xI_{n+1} + V_n, \qquad I_{n+1} = V_n + I_n.$$

Now assume that $I_n = B_{n-1}(x)$ and $V_n = b_n(x)$. Then,

$$V_{n+1} = xB_n(x) + b_n(x) = b_{n+1}(x),$$
$$I_{n+1} = b_n(x) + B_{n-1}(x) = B_n(x),$$

applying Lemma 1 to the expression for I_{n+1}, which completes the induction.

We note that

$$V_{n+1} = b_{n+1}(x) = x[B_n(x) + B_{n-1}(x) + \cdots + B_0(x) + 1];$$
$$B_n(x) = I_{n+1} = V_n + V_{n-1} + \cdots + V_0 = b_n(x) + b_{n-1}(x) + \cdots + b_0(x).$$

These follow directly from the special resistive network.

REFERENCES

1. A. M. Morgan-Voyce, "Ladder Network Analysis Using Fibonacci Numbers," IRE Transactions on Circuit Theory, Vol. CT-6, Sept. 1959, pp. 321-322.
2. S. L. Basin, "The Appearance of Fibonacci Numbers and the Q-matrix in Electrical Network Theory," Mathematics Magazine, Vol. 36, Mar.-Apr. 1963, pp. 84-97.
3. Richard A. Hayes, Fibonacci and Lucas Polynomials, Master's Thesis, San Jose State College, San Jose, Calif., 1965.
4. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, No. 1, Feb. 1966, pp. 73-81.
5. V. E. Hoggatt, Jr., Problem H-73, Fibonacci Quarterly, Vol. 3, No. 4, Dec. 1965; Solution by D. A. Lind, Vol. 5, No. 3, Oct. 1967, p. 256.
6. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII — An Introduction to Fibonacci Polynomials and their Divisibility Properties," Fibonacci Quarterly, Vol. 8, No. 4, Oct. 1970, pp. 407-420.
7. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5, Dec. 1969, pp. 457-463.
8. V. E. Hoggatt, Jr., and C. T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," Fibonacci Quarterly, Vol. 12, No. 2 (Apr. 1974), pp. 113-120.
9. V. E. Hoggatt, Jr., and Gerald Bergum, "Irreducibility of Lucas and Generalized Lucas Polynomials," Fibonacci Quarterly, Vol. 12, No. 1 (Feb. 1974), pp. 95-100.
10. M. N. S. Swamy, "Further Properties of Morgan-Voyce Polynomials," Fibonacci Quarterly, Vol. 6, No. 2, Apr. 1968, pp. 167-175.

(Continued from page 146.)

The material consists of two pages of explanation, six pages of tables for systematizing the work of finding the Fibonacci and Lucas expressions in parentheses, and 78 pages of formulas. There are 625 formulas in all arranged in categories according to the difference relation from which they are derived.

The material may be obtained by writing to the Managing Editor:

Brother Alfred Brousseau
St. Mary's College
Moraga, Calif. 94575

In loose-leaf form, the price is \$3.50; with ring binding and flexible cover, the price is \$4.00.