GENERATING IDENTITIES FOR PELL TRIPLES

CARL SERKLAND
San Jose State University, San Jose, California 95192

This paper is modelled after an article by Hansen [1] dealing with identities for Fibonacci and Lucas triples. Free use has been made of the methods of that article, and this paper follows its format closely. It is hoped that seeing Fibonacci methods used in a slightly different context will lead the reader to a deeper understanding of those methods, in addition to the production of some new Pell identities.

The Pell sequence is closely akin to the Fibonacci sequence; it is defined by $P_0 = 0$, $P_1 = 1$, $P_{n+2} = P_n + 2P_{n+1}$. This gives us the sequence 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, $\cdots$. We may also define a Pell analogue of the Lucas sequence: $R_0 = 2$, $R_1 = 2$, $R_{n+2} = R_n + 2R_{n+1}$. It is simple to show that, with these definitions, $P_{n+1} + P_{n-1} = R_n$. Another useful result, easily proved by the usual Fibonacci methods, gives the Pell sequence and its Lucas analogue as functions of their subscripts:

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad R_n = \alpha^n + \beta^n ,$$

where

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2} .$$

Note that α and β are roots of the equation $x^2 - 2x - 1 = 0$, and hence $\alpha\beta = -1$ and $\alpha + \beta = 2$.

Using the generating functions of

$$\{P_{n+m}\}_{n=0}^{\infty} \quad \text{and} \quad \{R_{n+m}\}_{n=0}^{\infty}$$

we shall obtain identities for the triples $P_pP_qP_r$, $P_pP_qR_r$, $P_pR_qR_r$, and $R_pR_qR_r$, where p, q, and r are fixed integers.

To derive the desired generating functions we note that, using the Binet form of the Pell numbers,

$$\sum_{n=0}^{\infty} P_{n+m}x^n = \sum_{n=0}^{\infty} \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} x^n$$

(1)

$$= \frac{1}{\alpha - \beta}\left(\alpha^m \sum_{n=0}^{\infty} \alpha^n x^n - \beta^m \sum_{n=0}^{\infty} \beta^n x^n\right)$$

$$= \frac{1}{\alpha - \beta}\left(\alpha^m \frac{1}{1 - \alpha x} - \beta^m \frac{1}{1 - \beta x}\right)$$

$$= \frac{1}{\alpha - \beta}\left(\frac{(\alpha^m - \beta^m) - \alpha\beta(\alpha^{m-1} - \beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)}\right)$$

$$= \frac{P_m - P_{m-1}x}{1 - 2x - x^2} .$$

In a similar fashion we find

(2)
$$\sum_{n=0}^{\infty} R_{n+m}x^n = \frac{R_m + R_{m-1}x}{1 - 2x - x^2} .$$

We now evaluate formulas (1) and (2) for $-2 \le m \le 4$, letting $1 - 2x - x^2 = D$.

$$\sum_{n=0}^{\infty} P_{n-2}x^n = \frac{P_{-2} + P_{-3}x}{D} = \frac{-2 + 5x}{D} ; \qquad \sum_{n=0}^{\infty} R_{n-2}x^n = \frac{R_{-2} + R_{-3}x}{D} = \frac{6 - 14x}{D}$$

$$\sum_{n=0}^{\infty} P_{n-1}x^n = \frac{P_{-1} + P_{-2}x}{D} = \frac{1 - 2x}{D} ; \qquad \sum_{n=0}^{\infty} R_{n-1}x^n = \frac{R_{-1} + R_{-2}x}{D} = \frac{-2 + 6x}{D}$$

$$\sum_{n=0}^{\infty} P_n x^n = \frac{P_0 + P_{-1}x}{D} = \frac{0 + x}{D} ; \qquad \sum_{n=0}^{\infty} R_n x^n = \frac{R_0 + R_{-1}x}{D} = \frac{2 - 2x}{D}$$

$$\sum_{n=0}^{\infty} P_{n+1}x^n = \frac{P_1 + P_0x}{D} = \frac{1}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+1}x^n = \frac{R_1 + R_0x}{D} = \frac{2 + 2x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+2}x^n = \frac{P_2 + P_1x}{D} = \frac{2 + x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+2}x^n = \frac{R_2 + R_1x}{D} = \frac{6 + 2x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+3}x^n = \frac{P_3 + P_2x}{D} = \frac{5 + 2x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+3}x^n = \frac{R_3 + R_2x}{D} = \frac{14 + 6x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+4}x^n = \frac{P_4 + P_3x}{D} = \frac{12 + 5x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+4}x^n = \frac{R_4 + R_3x}{D} = \frac{34 + 14x}{D} \quad .$$

Using the fact that two series are equal if and only if the corresponding coefficients are equal, we now find several elementary identities.

Since

$$\frac{2 - 2x}{D} = \frac{1}{D} + \frac{1 - 2x}{D} \quad ,$$

it follows that

$$\sum_{n=0}^{\infty} R_n x^n = \sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} P_{n-1}x^n$$

$$= \sum_{n=0}^{\infty} (P_{n+1} + P_{n-1})x^n$$

and hence

(3) $$R_n = P_{n+1} + P_{n-1} \, ; \qquad n \text{ a whole number} .$$

Using the Binet forms, it is not difficult to show that $P_{-n} = (-1)^{n+1}P_n$ and $R_{-n} = (-1)^n R_n$ for any positive integer n.

We now observe that

$$\begin{aligned} P_{(-n)+1} + P_{(-n)-1} &= P_{-(n-1)} + P_{-n(n+1)} \\ &= (-1)^{(n-1)+1} P_{n-1} + (-1)^{(n+1)+1} P_{n+1} \\ &= (-1)^n (P_{n-1} + P_{n+1}) \\ &= (-1)^n R_n \\ &= R_{-n} \quad . \end{aligned}$$

Hence Eq. (3) holds for all integers n.

We now proceed with some theorems necessary to the development of Pell triples.

<u>Theorem 1.</u> $P_n R_m + P_{n-1} R_{m-1} = R_{m+n-1}$.

<u>Proof.</u> Let m be any fixed integer. Then

$$\begin{aligned} \sum_{n=0}^{\infty} (P_n R_m + P_{n-1} R_{m-1}) x^n &= R_m \sum_{n=0}^{\infty} P_n x^n + R_{m-1} \sum_{n=0}^{\infty} P_{n-1} x^n \\ &= R_m \frac{x}{D} + R_{m-1} \left(\frac{1 - 2x}{D} \right) \\ &= \frac{R_m x + R_{m-1} - 2R_{m-1} x}{D} = \frac{R_{m-1} + R_{m-2} x}{D} \\ &= \sum_{n=0}^{\infty} \frac{R_{n+m-1}}{D} x^n \end{aligned}$$

and, equating summands,

$$P_n R_m + P_{n-1} R_{m-1} = R_{n+m-1} \quad .$$

<u>Theorem 2.</u> $P_n P_m + P_{n-1} P_{m-1} = P_{n+m-1}$.

<u>Proof.</u>

$$\begin{aligned} \sum_{n=0}^{\infty} (P_n P_m + P_{n-1} P_{m-1}) x^n &= P_m \sum_{n=0}^{\infty} P_n x^n + P_{m-1} \sum_{n=0}^{\infty} P_{n-1} x^n \\ &= P_m \frac{x}{D} + P_{m-1} \frac{1 - 2x}{D} \\ &= \frac{P_m x + P_{m-1} - 2P_{m-1} x}{D} = \frac{P_{m-2} x + P_{m-1}}{D} \\ &= \sum_{n=0}^{\infty} \frac{P_{n+m-1}}{D} x^n \quad , \end{aligned}$$

and, equating summands,

$$P_n P_m + P_{n-1} P_{m-1} = P_{n+m-1} .$$

Theorem 3. $R_n R_m + R_{n-1} R_{m-1} = R_{n+m} + R_{n+m-2} = 8 P_{n+m-1}$.

Proof. Let m be any fixed integer. Then

$$\sum_{n=0}^{\infty} (R_n R_m + R_{n-1} R_{m-1}) x^n$$

$$= R_m \sum_{n=0}^{\infty} R_n x^n + R_{m-1} \sum_{n=0}^{\infty} R_{n-1} x^n$$

$$= R_m \frac{2 - 2x}{D} + R_{m-1} \frac{-2 + 6x}{D}$$

$$= \frac{2R_m - 2R_m x - 2R_{m-1} + 6R_{m-1} x}{D}$$

$$= \frac{2(R_m - R_{m-1}) + 2(3R_{m-1} - R_m)x}{D}$$

$$= \frac{R_m + R_{m-2} + (2R_{m-1} - 2R_{m-2})x}{D}$$

$$= \frac{R_m + R_{m-2} + (R_{m-1} + R_{m-3})x}{D}$$

$$= \frac{R_m + R_{m-1} x + R_{m-2} + R_{m-3} x}{D}$$

$$= \sum_{n=0}^{\infty} R_{n+m} x^n + \sum_{n=0}^{\infty} R_{n+m-2} x^n$$

$$= \sum_{n=0}^{\infty} (R_{n+m} + R_{n+m-2}) x^n$$

and hence,

$$R_n R_m + R_{n-1} R_{m-1} = R_{n+m} + R_{n+m-2} .$$

Now,

$$\begin{aligned}R_{n-1} + R_{n-1} &= (P_{n+2} + P_n) + (P_n + P_{n-2})\\ &= 2P_{n+1} + 3P_n + P_{n-2}\\ &= 4P_n + 3P_n + 2P_{n-1} + P_{n-2}\\ &= 8P_n \quad .\end{aligned}$$

We now use a partial fractions technique to find the final necessary result:

$$\begin{aligned}\frac{(p + qx)}{D}\,\frac{(r + tx)}{D} &= \frac{pr + (pt + qr)x + qtx^2}{D^2}\\ &= \frac{-qt}{D} + \frac{(pr + qt) + (pt + qr - 2qt)x}{D^2}\end{aligned}$$

(4)
$$\begin{aligned}\frac{P_m + P_{m-1}x}{D}\cdot\frac{R_s + R_{s-1}x}{D} &= \sum_{n=0}^{\infty} P_{n+m}x^n \cdot \sum_{n=0}^{\infty} R_{n+s}x^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{k+m}R_{n-k+s}x^n\end{aligned}$$

but also, by Eq. (4),

$$\begin{aligned}\frac{P_m + P_{m-1}x}{D}\cdot\frac{R_s + R_{s-1}x}{D} &= \frac{-P_{m-1}R_{s-1}}{D}\\ &+ \frac{(P_mR_s + P_{m-1}R_{s-1}) + (P_mR_{s-1} + P_{m-1}R_s - 2P_{m-1}R_{s-1})x}{D^2}\\ &= \frac{-P_{m-1}R_{s-1}}{D} + \frac{R_{m+s-1} + (P_{m-1}R_s + P_{m-2}R_{s-1})x}{D^2}\\ &= (-P_{m-1}R_{s-1})\,\frac{1}{D} + \frac{R_{m+s-1} + (P_{m-1}R_s + P_{m-2}R_{s-1})x}{D^2}\\ &= -P_{m-1}R_{s-1}\sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} R_{n+m+s-1}x^n \sum_{n=0}^{\infty} P_{n+1}x^n\\ &= \sum_{n=0}^{\infty} (-P_{n+1}P_{m-1}R_{s-1})x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{k+1}R_{n-k+m+s-1}x^n\\ &= \sum_{n=0}^{\infty}\Big(P_{n+1}P_{m-1}R_{s-1} + \sum_{k=0}^{n} P_{k+1}R_{n-k+m+s-1}\Big)x^n \quad .\end{aligned}$$

Hence,

$$\sum_{k=0}^{n} P_{k+m} R_{n-k+s} = -P_{n+1} P_{m-1} R_{s-1} + \sum_{k=0}^{n} P_{k+1} R_{n-k+m+s-1}$$

and

$$P_{n+1} P_{m-1} R_{s-1} = \sum_{k=0}^{n} (P_{k+1} R_{n-k+m+s-1} - P_{k+m} R_{n-k+s}) .$$

Letting $p = m - 1$, $q = n + 1$, and $r = s - 1$, we obtain

Theorem 4.

$$P_p P_q R_r = \sum_{k=0}^{q-1} (P_{k+1} R_{p+q+r-k} + P_{p+k+1} R_{q+r-k}) .$$

Now we convolute

$$\frac{P_m + P_{m-1}x}{D} \quad \text{with} \quad \frac{P_t + P_{t-1}x}{D}$$

and, using the previous procedures, we find

Theorem 5.

$$P_p P_q P_r = \sum_{k=0}^{r-1} (P_{p+q+r-k} P_{k+1} - P_{p+k-1} P_{q+r-k}) .$$

Similarly, we convolute

$$\frac{R_m + R_{m-1}x}{D} \quad \text{with} \quad \frac{R_t + R_{t-1}x}{D}$$

to obtain

Theorem 6.

$$P_p R_q R_r = \sum_{k=0}^{p-1} (8P_{q+r+k+1} P_{p-k} - R_{q+k+1} R_{p+r-k}) .$$

Now,

$$\begin{aligned} R_p R_q R_r &= (P_{p+1} + P_{p-1}) R_q R_r \\ &= P_{p+1} R_q R_r + P_{p-1} R_q R_r \end{aligned}$$

$$R_pR_qR_r = \sum_{k=0}^{p}(8P_{q+r+k+1}P_{p-k+1} - R_{q+k+1}R_{p+r-k+1})$$

$$+\sum_{k=0}^{p-2}(8P_{q+r+k+1}P_{p-k-1} - R_{q+k+1}R_{p+r-k-1})$$

$$= \sum_{k=0}^{p-1}\Big[8P_{q+r+k+1}(P_{p-k} + P_{p-k-1})$$

$$- R_{q+k+1}(R_{p+r-k+1} + R_{p+r-k-1})\Big]$$

$$+ (8P_2P_{q+r+p} - R_{q+p}R_{r+2})$$

$$+ (8P_1P_{p+q+r+1} - R_{q+p+1}R_{r+1})$$

$$= \sum_{k=0}^{p-2} 8(P_{q+r+k+1}R_{p-k} - R_{q+k+1}P_{p+r-k})$$

$$+ 8(2P_{p+q+r} + P_{p+q+r+1})$$

$$- (R_{p+q}R_{r+2} + R_{p+q+1}R_{r+1})$$

$$= 8\sum_{k=0}^{p-2}(P_{q+r+k+1}R_{p-k} - R_{q+k+1}P_{p+r-k})$$

$$+ 8P_{p+q+r+2} - (2R_{p+q}R_{r+1} + R_{p+q}R_r + R_{p+q+1}R_{r-1})$$

and, by Theorem 3, we obtain

Theorem 7.

$$R_pR_qR_r = 8\left[\sum_{k=0}^{p-2}(P_{p+q+r+k+1}R_{p-k} - P_{p+r-k}R_{q+k+1}) - P_{p+q+r-1}\right] - 2R_{p+q}R_{r+1}\,.$$

REFERENCES

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2. Carl E. Serkland, "The Pell Sequence and Some Generalizations," Master's Thesis, California State University, San Jose, California, January 1973.